Illustrations 5 and 7 (pages 49 and 52 of the original).

Coordinates Φ and Λ and angles α and β of a shape-true circle-grid.

(n = 0.7)

[See above noted table on page 107a following.]

The grid is without doubt a better reproduction of the entire earth than Lambert's grid with n = 0.5, even than the all-circular with n = 1 and the one after Lagrange with $a = \sqrt{2}$, both of which need the infinite plane or even more yet than that for the reproduction.

One can also make the stipulation that the entire length of the central-meridian, as compared with the entire length of the equator, be decreased at the same ratio as that of the border meridian is increased. This ratio would result in 1.4303; and one would have to employ n = 0.6115. For this projection, too, (SNo. 172) the little table above, "Shape-true Circle-grids," gives a view of the process of Φ and Λ.

In the last sort of this class, the pictures of the primary-circles $\lambda = \pm \pi/2$ are two equal-spaced semi-circles, as in the projections No. 158 (globular for the hemisphere) and No. 170 (all-circular, transverse-axical). In the type of this sort, we let the mean of Φ and/or Λ of the two named projections apply for Φ and/or Λ. The thus produced combination map (SNo. 174) is Nell's modified globular projection (1852).

TEXT NOT REPRODUCIBLE

Koordinaten Φ und Λ und Winkel β und α eines winkeltreuen Kreisnetzes

($n = 0,7$)

$\varphi°$	$\beta°$	Φ	$\lambda°$	$\alpha°$	Λ	$\lambda°$	$\alpha°$	Λ
15	16° 33,7′	0,0523	15	10,5°	0,0219	105	73,5°	0,7467
30	21 30,4	0,1000	30	21,0	0,1553	120	84,0	0,9001
45	33 15,0	0,2901	45	31,5	0,2320	135	94,5	1,0318
60	46 37,0	0,4007	60	42,0	0,3830	150	105,0	1,3022
75	62 45,3	0,6105	75	52,5	0,4932	165	115,5	1,5849
90	90 0,0	1,0000	90	63,0	0,6128	180	126,0	1,9326

Polar-point maps with non-circle-shaped primary lines (SNo.175-179)

The H are not circles in the following class. Until now, only such projections of a subclass with scale-true central H have been submitted. The basic-line is also scale-true in the first group. In the first sort, the pictures of the primary-circles $\lambda = \pm \pi/2$ are again two equal-spaced semi-circles, as in projections SNo. 158, 170, 174, so that the N are determined as arcs through the three dividing-points on the H $= -\pi/2$, 0, $+\pi/2$. In the type (SNo. 175), the H are determined as the same half-ellipses as in the projection by Mollweide (No. 136). No. 175 is projection I by Fournier (1646).

The second sort includes the *all-conical* projections. The first type (No. 176) has its H so determined that all all-conical determined N should be scale-true. The equations are: $\Phi = \phi$; $r = \cot \phi$ and $\beta = \lambda \sin \phi$. This is the ordinary *polyconical projection* rendered in 1855 by the *American Coast and Geodetic Survey*. The H of the second type (No. 177) are so determined that all N intersect rectangularly. The equations of this projection read $\Phi = \phi$; $r = \cot \phi$; $\operatorname{tg} \beta/2 = \lambda/2 \sin \phi$. This is the *rectangular polyconical projection of the British War Office* given in 1860.

Projection No. 178, which one may call *secondary-circular* all-globular, also belongs with the group with scale-true basic-lines. In this projection, the border-meridians of the full-globe and the N are drawn exactly as in the full-globular projection (No. 159). The other H, however, are not drawn as circles, but so that all N are equal spaced. This projection is just a little different from No. 159. Its equations are:

$$x = \frac{5\pi}{8} \sin \phi; \quad y_\pi = \pm \frac{\pi}{8} (3 + 5 \cos\phi);$$

$$tg \frac{\beta_\pi}{2} = \frac{\phi - x_\pi}{y_\pi}; \qquad \beta(\lambda) = \frac{\lambda}{\pi}\beta_\pi$$

One may foresee a sort with *all-conical* projections also in the next group with *non-scale-true* basic-lines. As type for it, I have derived an area-true, all-circular projection (with scale-true central H) on the basis of the general theory of area-true, row-circular projections given in Tissot-Hammer[71] (SNo. 179), and I have drawn its grid (Illustration 10, plate V). According to Tissot-Hammer (and according to Herz), the equation

$$\frac{r}{\sin \delta} \left[\frac{dr}{d\delta} \beta - \frac{ds}{d\delta} \sin\beta \right] = \lambda + t$$

applies for all area-true, row-circular projections, where the designations of our Illustration 5 (plate I) correspond to, and t is an arbitrary function of $\delta = (90° - \phi)$. In illustration 5, M is the picture-point of the global-point (λ, δ) on the N-circle with the radius r at the point S, which stands off from the basic-line at s = OS and β = OSB. If we take l = 0 and employ $r = tg\,\delta$; $s = \frac{\pi}{2} + tg\,\delta - \delta$ as must be done with the all-conical projections, then the above true-area stipulation for the determination of β yields the equations

$$\beta = \sin \beta \sin^2\delta + \lambda \cos^3 \delta \text{ or } (1)\ \beta = \sin \beta \cos^2\phi + \lambda \sin^3\phi$$

and the coordinates x, y of the point M are: (2) $x = \phi + 2\cot\phi\sin^2\frac{\beta}{2}$
$y = \cot \phi \sin \beta$ (3)

$\beta = \pm\lambda$ and $dy/dx = -tg\lambda$ for $\phi = \pm\ \pi/2$. The projection is thus shape-true at the two polar-points $x = \phi = \pm\ \pi/2$. $x = 0$ and $\beta = 0$ for the points of the basic-line $\phi = 0$. The scale y on the basic-line cannot be directly calculated from the equation $y = \sin\beta / tg\ \phi = 0/0$. In a letter to me, however, F. Lange derived that for $\phi = 0$ one can put the equation for y in the form $y^3 + 6y = 6\lambda$. The derivation is as follows: in $y = \sin\ \beta/tg\ \phi$, with nominator and denominator differentiated according to ϕ, yields

$$y = \cos^2\phi\ \cos\beta\ \frac{\delta\beta}{\delta\phi} \quad \text{thus for } \phi = \beta = 0;\ y = \frac{\delta\beta}{\delta\phi} = \beta'$$

Also $y = \sin\beta/\sin\phi$ can be employed for $\phi = \beta = 0$ in the place of $y = \sin{}^\beta / tg{}^\phi$; thus, for ${}^\phi = {}^\beta = 0$, $y = \sin{}^\beta / \sin{}^\phi = \beta'$; β' can also be obtained according to ϕ by means of differentiation of the equation (1). This yields

$$\beta' = \frac{3\ \lambda\ \sin^3\phi\ \cos\ \phi - \sin\ 2\phi\ \sin\ \beta}{1 - \cos\ \beta\ \cos^2\phi} = \frac{0}{0}$$

for $\phi = \beta = 0$

By means of differentiation of nominator and denominator according to ϕ and cancellation with $\sin\phi\ \cos^2\phi$, one finds:

$$\beta' = \frac{6\lambda - 3\lambda\ tg^2\phi - 2\ \frac{\cos\ 2\phi}{\cos^2\phi}\ \frac{\sin\beta}{\sin\phi} - 2\frac{\cos\ \beta}{\cos\ \phi}\ \beta'}{\frac{\sin\ \beta}{\sin\ \phi}\ \beta' + 2\ \frac{\cos\ \beta}{\cos\ \phi}}$$

that is, for $\phi = \beta = 0$, if one employs y for $\sin\beta/\sin\phi$ and for β':

from which y as function λ can be calculated also on the basic-line $\phi = 0$.

In the table on the next page, the number values r, s, β x and y for such an area-true all-conical projection are compiled in a selection which suffices for the drawing of a grid of 15° to 15° in longitude and latitude.

The grid of this projection, corresponding to a globe diameter of 75 mm, is reproduced in illustration 10 (plate V). In this projection, it looks as though the two H $\lambda = \pm 90°$ had come together so as to form an ellipse. But that is not the case. One cannot obtain an equation, from the three determining equations (1), (2), (3) of the projection, by means of eliminating β and ϕ, which represents an ellipse equation for $\lambda = \pm \pi/2$ in x and y. Our H bends somewhat more sharply outward than the ellipse through the four axial points $\phi = 0$, $\lambda = \pm \pi/2$ and $\lambda = 0$, $\phi = \pm \pi/2$. This interesting projection combines circle-shaped parallels of latitude with true-area, and true-scale and true-shape on the central-meridian without stretching a single point out to the line. It is unfortunate, however, that in this projection the equator is about 5/4 or so long as the central meridian, rather than being two times as long. An essentially more fortunate transformation of this projection is rendered in SNo. 184.

Polar-line maps with non-circle-shaped primary-lines (SNo.180,181)

Polar line maps as in our subbranch A, in which no two N intersect each other, have up to now never been proposed. The meridianal stretching in the polar areas of those circle-grids of class I--which reproduce the earth as full circle (SNo. 166, 167, 173)--incited one to correct this deficiency by transforming the figures of the ϕ-scale in such a grid and letting some N-circle act as picture of the pole, thus changing over to a polar-line map. As an example of this, I have carried it out in SNo. 180

[The following table is found on page 112a.]

Erlauterung der Systemtabelle = Explanation of the system table

Koordinaten des flachentreuen allkegeligen Entwurfs (SNr. 179) = Coordinates of the area-true, all-conical projection (SNo. 179)

German decimal point is expressed: ","

with the *rectangular circle-grid* , which is the basis of the earth map no. II (SNo. 167) by van der Grinten. Furthermore, I have made the central-H scale-true. To do this, one only needs to rewrite that determining equation of projection 167 for the scale on the central-H, $\Phi = \pi \text{ tg } {}^{\mu}/_{2}$, so that it takes on the form of $\phi = \pi \text{ tg } {}^{\mu}/2$, and to then use this equation, in the place of the now discarded equation $\sin\mu = 2\phi : \pi$ for the calculation of μ as function of ϕ, while the equations $r = \pi \cot\mu$ and $\rho = (\pi^2 + \lambda^2) : (2\lambda)$ remain unchanged. The grid (Illustration II, plate II)

Erläuterung der Systemtabelle
Explanation of the system table

Coordinates of the area-true, all-conical projection (SNo. 179)

Koordinaten des flächentreuen allkegeligen Entwurfs (SNr. 179)

(Für $\varphi = 0$ ist $\beta = 0$; für $\varphi = \frac{\pi}{2}$ ist $s = x = \frac{\pi}{2}$). Kugelradius = 1
For / For / Radius of globe

$\lambda°$	$\varphi°$	0°	7,5°	15°	22,5°	30°	37,5°	45°	52,5°	60°	70°	75°	80°
	r	∞	7,5958	3,7321		1,7321		1,0000		0,5774	0,3640	0,2679	0,1763
	s	∞	7,7267	3,0039		2,2557		1,7854		1,6249	1,5857	1,5760	1,5723
0°	y	0	0,1309	0,2618		0,5236		0,7854		1,0472	1,2217	1,3090	1,3963
15	β	0	1° 56,16′	3° 50,4′		7° 26,22′		10° 32,32′		12° 57,18′	14° 4,02′	14° 23,6′	
"	x	0	0,1252	0,2704		0,5382		0,8923		1,0612	1,2327	1,3175	
"	y	0,2573	0,2566	0,5499		0,2242		0,18304		0,1294	0,0885	0,0670	
30	β	0	3° 45,6′	7° 28,0′		14° 32,0′		20° 45,72′		25° 41,76′	28° 2,54′	28° 53,31′	
"	x	0	0,1472	0,2034		0,5790		0,8503		1,1043	1,2642	1,3423	
"	y	0,5207	0,4931	0,4950		0,4346		0,3545		0,2504	0,1711	0,1294	
45	β	0		10° 45,6′		21° 4,8′		30° 24,6′		33° 3,43′	41° 43,48′	43° 10,68′	44° 11,04′
"	x	0		0,3274		0,6365		0,9230		1,1709	1,3144	1,3916	1,4462
"	y	0,7187		0,6068		0,6239		0,5052		0,3559	0,2426	0,1834	0,1220
60	β	0	6° 52,8′	13° 42,0′		27° 1,5′		39° 23,7′		49° 55,9′	55° 17,82′	57° 17,33′	
"	x	0	0,1856	0,3680		0,7027		1,0124		1,2529	1,3785	1,4322	
"	y	0,9214	0,9009	0,8300		0,7870		0,6347		0,4418	0,2932	0,2255	
75	β	0	8° 10,8′	16° 19,8′		32° 24,0′		47° 42,49′		61° 10,38′	68° 28,14′	71° 13,14′	
"	x	0	0,2052	0,4121		0,7032		1,1125		1,3479	1,4521	1,4906	
"	y	1,0967	1,0809	1,0493		0,9231		0,7307		0,5063	0,3465	0,2537	
90	β	0		18° 42,0′	28° 0,9′	37° 16,5′	48° 26,1′	55° 24,0′	64° 1,92′	72° 4,8′	81° 18,0′	84° 55,8′	87° 41,4′
"	x	0		0,4588	0,6756	0,8773	1,0596	1,2176	1,3477	1,4469	1,5306	1,5503	1,5855
"	y	1,2114		1,1969	1,1840	1,0490	0,9443	0,8231	0,6899	0,5493	0,3598	0,2660	0,1762
105	β	0	10° 24,0′	20° 49,8′		41° 43,5′		62° 32,34′		82° 23,7′	93° 49,9′	98° 24,90′	
"	x	0	0,2557	0,5057		0,9630		1,3243		1,5481	1,6097	1,6162	
"	y	1,3916	1,3712	1,3271		1,1528		0,8874		0,5723	0,3692	0,2651	
120	β	0	11° 22,9′	22° 46,2′	34° 15,0′	45° 48,4′	57° 29,86′	69° 12,6′	80° 58,4′	92° 44,82′	106° 0,96′	111° 42,18′	
"	x	0	0,2901	0,5526	0,8114	1,0483	1,2439	1,4304	1,5621	1,6522	1,6868	1,6760	
"	y	1,5087	1,4988	1,4444	1,3538	1,2419	1,0989	0,9349	0,7577	0,5769	0,3499	0,2490	
135	β	0		24° 33,6′		49° 36,0′		75° 26,4′		101° 42,0′	117° 56,4′	124° 48,49′	130° 15,48′
"	x	0		0,5955		1,1730		1,5344		1,7414	1,7562	1,7209	1,6865
"	y	1,6235		1,5512		1,3191		0,9689		0,5652	0,3215	0,2200	0,1345
150	β	0	13° 3,9′	26° 12,6′	39° 32,4′	53° 7,5′	67° 3,00′	81° 21,42′	96° 1,20′	110° 48,54′	129° 37,74′	137° 45,08′	
"	x	0	0,3271	0,6455	0,9451	1,2163	1,4496	1,6409	1,7641	1,8296	1,8179	1,7721	
"	y	1,7234	1,7152	1,6482	1,5370	1,3855	1,2001	0,9846	0,7651	0,5397	0,2893	0,1802	
165	β	0	13° 54,0′	27° 45,3′		56° 25,8′		86° 59,7′		119° 36,5′	141° 7,2′	150° 34,42′	
"	x	0	0,3533	0,6912		1,2970		1,7330		1,9127	1,8690	1,8193	
"	y	1,8337	1,9247	1,7382		1,4432		0,9936		0,5029	0,2295	0,1316	
180	β	0		29° 12,0′	44° 0,6′	59° 33,0′	75° 31,62′	92° 15,9′	109° 51,3′	128° 10,2′	152° 27,6′	163° 18,0′	172° 9,42′
"	x	0		0,7253	1,0750	1,3789	1,6320	1,9249	1,9447	1,9913	1,9084	1,8336	1,7473
"	y	1,9347		1,8270	1,6919	1,4931	1,2619	0,9992	0,7217	0,4539	0,1693	0,0770	0,0241

S

TEXT NOT REPRODUCIBLE

German decimal point is expressed: ","

is, then, extremely easy to calculate and comfortable to draw. The following table gives some values for r and ρ at equator degrees as functions of ϕ and/or λ :

Rectangular Circle-Grid with Scale-true Central-H and Basic-Line (SNo. 180)

[The above noted table may be found on page 113a following.]

Illustration 11 shows the quite nice looking rectangular circle-grid, in which the rest of the meridians and parallels of latitude appear almost spaced in equal form. The only disadvantage of this grid is that the poles are not reproduced as points, but as arcs with a length of 250.4 equator-degrees, whereas the polar lines are 360 equator degrees long on the likewise rectangular cylindrical world map, and the rectangularity is missing on the Eckert world maps with polar line length of 180 equator degrees.

The next subbranch B also offers polar lines where all N are arcs through the two points $\phi = 0$, $\lambda = \pm\pi$. These are polar-line maps, circle-grids of the second class, which are discussed in Tissot-Hammer[72]. These grids are the same as those of the first class (SNo. 158 - 174); except that now the N-lines $\phi = 0$ to $\phi = \pm 90$ are considered as H-lines from $\lambda = 0$ to $\lambda = \pm 180°$, and inversely, the H from $\lambda = 0$ to $\lambda = \pm 180°$ are now considered as N from $\phi = 0$ to $\phi = \pm 90°$. Such maps can only be useful for representing smaller parts of the globe. They cannot be used

Rechtschnittiges Kreisnetz mit maßtreuer Mittel-H und Grundlinie (SNr. 180)

$\Phi = \varphi$	r	$\Lambda = \lambda$	ϱ	$\Lambda = \lambda$	ϱ
15°	1072,70°	15°	1060,80°	105°	206,79°
30	524,92	30	555,00	120	195,00
45	337,46	45	382,50	135	187,50
60	240,00	60	300,00	150	183,00
75	178,50	75	253,50	165	180,72
90	135,00	90	225,00	180	180,00

as world maps, since the poles are stretched out into lines of far more than equator length, and, furthermore, they let the border-meridians $\lambda = \pm 180°$ shrink down to points, when the circle-grid of the first class is a point map, or, when it is a polar-line map, to a small arc. An example of the latter type is given in Tissot-Hammer (fig. 17), a shape-true circle-grid of the second class, for which the following equations apply:

$$\rho = \cot\alpha; \quad \Lambda = \operatorname{cosec}\alpha - \cot\alpha, \text{ where log nat } \cot\frac{\alpha}{2} = \lambda$$

$$r = \operatorname{cosec}\beta; \quad \Phi = \operatorname{cosec}\beta - \cot\beta, \text{ where } \beta = \text{log nat } \cot\frac{\delta}{2};$$

$$\delta = \frac{\pi}{2} - \phi$$

In this grid, SNo. 181, the parallels of latitude are arcs through the points $x = 0$; $y = \pm 1$. The map, expanded to the latitudes $= \pm 85$ 3.08', fills up the whole plane, with the exception of two small circles with radius 0.0866 at the points $x = 0$; $y = \pm 1.0037$. The small circles are the pictures of the meridians $\lambda = \pm 180$, whereas the meridian $\lambda = 0$ is the infinitely long straight line $y = 0$. The equator has an entire length of 1.8266. The rest of the circles of latitude keep getting greater with the increasing latitude, until the circle of latitude ± 85 3.08' consists of the infinitely distant straight line and the straight line $x = 0$, with exception of the piece $81.0037 < y < + 1.0037$. In the grid by Tissot-Hammer drawn only up to $\pm 75°$

latitude, also, 75° on the central-meridian require more than double the room required by 90° on the equator.

FAMILY III, BRANCH B: CURVED-SYMMETRICAL, NOT ROW-CIRCULAR (SNo. 182 - 196)

Subbranch A: Transformed, Order a: Expanded (SNo. 182 - 184)

In Branch B, the N are not a system of circles, but are curves among which in single instances a circle may appear. The subbranch A contains projections, which are produced by means of *transformation* of already discussed doubly-symmetrical grids. It was already mentioned on p. 13 that the grid of the map, central-point in the transverse-axical central-perspective projection (SNo. 1), corresponds to the characteristics of family I (true-circular), but that the polar-grid belongs with family III (curved-symmetrical). The same applies for every other radial transverse-axical projection. They may all, according to their polar grid, be considered as members of family III. They and all projections of families II and III may be thought of as original forms which can be transformed.

The *first order* of these transformed projections is an *affine reproduction*, in which the ratio of the coordinates x : y changes uniformly for each map-point, the map thus being expanded or even compressed to various degrees in the x- and y-directions. The geometric order of all curves in such a transformation remains unchanged. All straight lines of the original map are also straight lines in the expanded map; all circles remain curves of the second order (cone-section) and in general become ellipses. Thus, if the polar grid of *transverse-axical radial* projections is expanded in the *first class* , then the great circles of the map center remain straight lines and their secondary-circles become conic sections. If there is an expansion of the original map in one direction in the same ratio as a compression in the other direction, then the area sizes remain unchanged, so that a new area-true projection is produced from an area-true original projection. However, true-shape is not maintained in the expansion, but the radial transverse-axical projections which are shape-true in the center do get two shape-true points rather than just this one.

It is for just this reason that the expanded transformation (SNo. 182) of the transverse-axical center-perspective projection (SNo. 1) is of significance. Because it has gained straight lines, the new projection is also *straight-directional* ; and since it now has two shape-true points, it is valuable, e.g., as radio directions finding map with two preferred radar stations. If the equations $x = \text{tg}\,\phi \sec\lambda$; $y = \text{tg}\,\lambda$ apply for the transverse-axical projection SNo. 1, then $x_1 = m\,\text{tg}\,\phi \sec\lambda$; $y_1 = n\,\text{tg}\,\lambda$ applies for the expanded transformation. The meridians remain parallel straight lines, the circles of latitude remain hyperbolae. True-shape is produced,

if $m>n$, at the central-meridian at the two points $x_1 = 0$; $y_1 = \pm\sin L$,

where $\cos L = n : M$

if $m<n$, " " equator " " " $y_1 = 0$; $x_1 = \pm\sin L$,

where $\cos L = m : n$

Naturally, true-shape can be obtained in such a projection also for two arbitrary map-points A and B; one has only to use an oblique-axical center-perspective projection as original, where the dividing point of arc A B is the map center, while the great circle A B takes the place of the equator. Herle drew the first straight-directional non-center-perspective projection [the designation "gnomonic" is not appropriate for straight-directional projections with two shape-true points] in the 1890's; merely in order to save space , he compressed the meridians of a transverse-axical gnomonic map at the same ratio. He did not realize that he had obtained two shape-true points in this manner. Littlehales [73] came up with a second such map in 1899. That other straight-directional maps with two shape-true points and more advantageous distortion are produced by means of collinear reproduction of gnomonic maps was shown by Maurer [74] in 1914; at this time he published a map of this nature. Further literature: Wedemeyer [75], Immler [76], Thorade [77] and Maurer [78].

As example of an area-true projection of this class, we may consider the not yet proposed expansion of the polar-grid of a transverse-axical area-true radial projection by Lambert (SNo. 23), because this new projection (SNo. 183) exhibits true-area with two shape-true points, and, under certain circumstances, far more advantageous distortion ratios than the--quite justifiedly--much used Lambert Projection. The equations $x = 2 \sin \delta/2 \cos \alpha$; $y = 2 \sin \delta/2 \sin \alpha$ apply for its polar grid, where δ is the arc-interval of the global-point from the prime-point, thus $\cos\ \delta = \cos\phi\cos\lambda$, and is the azimuth of the global point from the prime-point, thus $\operatorname{tg}\alpha = \sin\lambda\cot\ \phi$. In the expanded map, which should remain area-true, $x_1 = m\ x$ and $y_1 = y : m$. For the greatest shape-distortion 2 w,

at each equator-point, $\sin w = \left[m^2 - \cos^2\frac{\lambda}{2}\right] : \left[m^2 + \cos^2\frac{\lambda}{2}\right]$

at each central-meridian-point, $\sin w =$

$$\left[1 - m^2\cos^2\frac{\phi}{2}\right] : \left[1 + m^2\cos^2\frac{\phi}{2}\right]$$

For SNo. 23, $m = 1$. That more favorable distortion occurs for $m > 1$, has been shown on a map of Africa, according to both projections. Such a Lambert Projection can be found, for example, in Andree's Hand Atlas, 8th edition, 1928. The map extends at the central-meridian (20° east) from about 35° north to 35° south; and, at the equator, which passes through Africa from 9° east to 43° east, the uppermost shape-distortion 2 w to 28° difference in latitude is checked from the central-meridian. The following table for 2 w shows that an expanded map ($m = 1.0086$) does an overall better job for the area than the Lambert Projection with $m = 1$.

Uppermost Angle Distortion 2 w for Two Area-true Maps of Africa; Lambert ($m = 1$), expanded ($m = 1.0086$)

[The above noted table may be found on page 120a.]

On the expanded map the values of 2 w at the central-meridian-ends and the equator-ends are equal (4.4°) and less than at the meridians on the Lambert Map (5.4°). The central-meridian on the expanded map has two shape-true points ($\phi = \pm 15°$), between which the angle error remains under 1°.

When *row-circular projections are expanded*, the N-circles become ellipses. I have derived such a projection in SNo. 184, an area-true transformation of the all-conical, area-true projection No. 179. When one so freely gives up the characteristics of the all-conical, one obtains, then, two shape-true points $\lambda = 0$ on the equator, rather than one, and can strongly reduce the distortion of the great forms by letting the entire length of the equator become double so long as the central-meridian. *Illustration 12* (plate IV) shows this rather pleasing grid of an area-true world map with polar points, equal-spaced central meridian, elliptical parallels of latitude and two shape-true points at $\lambda = \pm 46.3°$ on the equator, which is double so long as the central-meridian. The equations for this projection are the same as those for SNo. 179, except for the factor n for y and 1:n for x, whereby n^2 must be twice the ratio of the lengths of the central-meridian : equator of the projection No. 179, 120a

Höchstwinkelverzerrungen 2 w für zwei flächentreue Karten von Afrika

Lambert ($m = 1$), gedehnt ($m = 1{,}0086$)

φ	λ	$m = 1$	$m = 1{,}0086$
0°	8° W u. 49° O	3° 27,4′	4° 26,4′
0	5° O u. 35° O	0 50,0	1 58,1
0	20° O	0 0,0	0 59,1
15° N u. 15° S	20° O	0 59,0	0 0,0
35° N u. 35° S	20° O	5 25,5	4 26,6

that is, $n^2 = 2 \cdot 1.5708 : 1.9347$ or $n = 1.2743$. The equations for No. 184 are thus:

(1) $\beta = \sin\beta\cos^2\phi + \lambda\sin^3\phi$; (2) $x = \frac{1}{n}\left[\phi + 2\cot\phi\sin^2\frac{\beta}{2}\right]$

(3) $y = n\cot\phi\sin\beta$; $n = 1.2743$.

The following table gives the coordinates according to the formulae for x and y. β has the same values as in the table for projection SNo. 179. $x = \frac{\pi}{2} : 1.2743 = 1.2327$ for $\phi = \frac{\pi}{2}$, that is, half as great as y for $\phi = 0°$; $\lambda = 180°$.

Coordinates of the Area-true Projection No. 184 (expanded, from No. 179): [See page 121a following.]

For finding the shape-true points on the equator, we use the way of thinking of F. Lange for deriving the scale-distribution on the equator of the original projection No. 179. In place of the equation there, $y^3 + 6y = 6\lambda$, we have for No. 184 the equation $\left(\frac{y}{n}\right)^3 + 6\frac{y}{n} = \lambda$

True-shape exists on the equator there, where, for $x = y = \beta = 0$,
$\frac{\delta\lambda}{\delta y}\ (\phi = 0) = \frac{\delta x}{\delta\phi}\ (\lambda = \text{constant})$
For $\phi = \beta = 0$, now follows from equation (3): $\frac{y}{n} = \frac{\sin\beta}{\sin\phi} = \frac{0}{0}$, and for $\phi = \beta = 0$ by means of differentiating denominator and nominator according to ϕ: $\frac{y}{n} = \frac{\delta\beta}{\delta\phi}$.

Differentiating from equation (3) yields

$$\frac{\delta y}{\delta\lambda} = n\cot\phi\cos\beta\frac{\delta\beta}{\delta\lambda}$$

TEXT NOT REPRODUCIBLE

Koordinaten des flächentreuen Entwurfs Nr. 184 (gedehnt aus Nr. 179)

λ		$\varphi = 0°$	7,5°	15°	22,5°	30°	37,5°	45°	52,5°	60°	70°	75°	80°
0	x	0	0,1027	0,2054		0,4109		0,6164		0,8218	0,9257	1,0263	1,0258
15	x	0	0,1095	0,2122		0,4223		0,6165		0,8331	0,9415	1,0340	
"	y	0,3279	0,3270	0,3181		0,2857		0,2322		0,1659	0,1125	0,0654	
30	x	0	0,1192	0,2302		0,4544		0,6673		0,8670	0,9921	1,0531	
"	y	0,6636	0,6217	0,6180		0,5538		0,4516		0,3205	0,2181	0,1650	
45	x	0		0,2569		0,5019		0,7241		0,9181	1,0315	1,0343	1,1319
"	y	0,9159		0,8878		0,7949		0,6459		0,4536	0,3051	0,2336	0,1506
60	x	0	0,1156	0,2888		0,5515		0,7941		0,9852	1,0817	1,1239	
"	y	1,1742	1,1595	1,1263		1,0028		0,8276		0,5630	0,3813	0,2873	
75	x	0	0,1631	0,3236		0,6298		0,8739		1,0572	1,1196	1,1698	
"	y	1,3818	1,3772	1,3371		1,1825		0,9423		0,6123	0,4115	0,3233	
90	x	0		0,3601	0,5302	0,6938	0,8315	0,9555	1,0576	1,1359	1,2017	1,2189	1,2256
"	y	1,5919		1,5348	1,4146	1,3367	1,2033	1,0183	0,5791	0,7000	0,4595	0,3524	0,2245
105	x	0	0,2036	0,3969		0,7557		1,0393		1,2149	1,2633	1,2683	
"	y	1,7604	1,7472	1,6912		1,4690		1,1363		0,7292	0,4626	0,3378	
120	x	0	0,2260	0,4336	0,6267	0,8226	0,9732	1,1226	1,2259	1,2966	1,3236	1,3153	
"	y	1,9225	1,9006	1,8407	1,7315	1,5825	1,4093	1,1914	0,9411	0,7322	0,4457	0,3211	
135	x	0		0,4685		0,8892		1,2012		1,3866	1,3782	1,3575	1,3235
"	y	2,0718		1,9767		1,6803		1,2335		0,7153	0,4037	0,2804	0,1715
150	x	0	0,2567	0,5037	0,7417	0,9516	1,1375	1,2577	1,3844	1,4556	1,4264	1,3906	
"	y	2,2050	2,1956	2,0991	1,9585	1,7656	1,5293	1,2531	0,9721	0,6878	0,3572	0,2256	
165	x	0	0,2775	0,5424		1,0155		1,3590		1,5010	1,4673	1,4206	
"	y	2,3307	2,3252	2,2128		1,8391		1,2726		0,6368	0,2911	0,1678	
180	x	0		0,5602	0,8155	1,0823	1,2808	1,4321	1,5241	1,5518	1,4976	1,4359	1,3712
"	y	2,4651		2,3201	2,1483	1,9026	1,6050	1,2733	0,9156	0,5791	0,2141	0,0931	0,0307

to which is added from equation (1) $\frac{\delta\beta}{\delta\lambda} = \frac{\sin^3\phi}{1 - \cos\beta\cos^2\phi}$ -

so that from both equations the result is:

$$\frac{\delta y}{\delta\lambda} = \frac{n^2 \sin^2\phi}{1 - \cos\beta\cos^2\phi} = \frac{0}{0} \qquad \text{for} \quad \phi = \beta = 0$$

By means of differentiating of nominator and demonimator according to ϕ and cancellation by $\sin\phi\cos\phi$, one obtains

$$\frac{\delta y}{\delta\lambda}(\phi = \lambda) = \frac{2n}{2 + \frac{\sin\beta}{\sin\phi}\frac{\delta\beta}{\delta\phi}} = \frac{2n^3}{n^2 + y^2}, \qquad \frac{\sin\beta}{\sin\phi} = \frac{\delta\beta}{\delta\phi} = \frac{y}{n} \quad \text{for } \phi=\beta=0$$

In order to obtain $\frac{\delta x}{\delta\phi}$, we differentiate equation (2), which we write in the form $x = \frac{\phi}{n} + \frac{y}{n}\,\mathrm{tg}\,\frac{\beta}{2}$ according to ϕ and obtain

$$\frac{\delta x}{\delta\phi} = \frac{1}{n} + \frac{1}{n^2}\frac{\delta y}{\delta\phi}\,\mathrm{tg}\,\frac{\beta}{2} + \frac{y}{2n^2\cos^2\beta}\frac{\delta\beta}{\delta\phi},$$ which results for

$\phi = \beta = 0$ in: $\frac{\delta x}{\delta\phi} = \frac{1}{n} = \frac{y^2}{2n^3} = \frac{2n^2 + y^2}{2n^3}$

True-shape exists for $\frac{\delta x}{\delta\phi} = \frac{\delta y}{\delta\lambda}$, that is: $\frac{2n^2 + y^2}{2n^3} = \frac{2n^3}{n^2 + y^2}$
from which results $\frac{y}{n} = \sqrt{2(n - 1)}$. If we employ this value of in the equations which applies for the equator,

$\left(\frac{y}{n}\right)^3 - 6\frac{y}{n} = 6\lambda$, then one finds for λ_w of the shape-true points

$\lambda_w = \frac{n + 2}{3}\sqrt{2(n - 1)}$, which with $n = 1.2743$ yeilds

$\lambda_w = \pm 0.8084 = \pm 46°19'$.

If one compares the reproductions 10 and 12 with the two area-true projections 179 and 184, then one sees that rectangular intersection of the N and H in No. 179 beyond the basic-line occurs only on the central-H, but in No. 184 beyond the central-H at least at higher latitudes, for the two values

$\pm\lambda_r$, too, which are different from 0. By elimination of from equations (2) and (3), one finds for the N-line the ellipse-equation:

$$\left(\frac{y}{n \cot}\right)^2 + \left(\frac{x - \frac{\phi + \operatorname{tg}\phi}{n}}{\frac{\cot\phi}{n}}\right)^2 = 1$$

from this, one can, by comparison with the usual ellipse-equation $\left(\frac{y}{b}\right) + \left(\frac{x - c}{a}\right) = 1$ read off the semi-axes a and b as well as the inclination of the ellipse-center $y = 0$; $x = c$. By employing the values a b c in the well-known equations for the tangent-direction at the ellipse-point (x,y),

$$\frac{dy}{dx} = -\frac{b}{a}\,\frac{(x - c)}{\sqrt{a^2 - (x - c)^2}}$$

also results. This $\frac{dy}{dx}$ must be, at places of rectangular intersection of H and N, the reciprocal value of that $\frac{dy}{dx}$ or $\left(\frac{dy}{d\phi} : \frac{dx}{d\phi}\right)$, which one finds for the meridian-equation λ ; that is, by differentiating the equations (2) and (3) according to ϕ with constant λ, while β applies as function of ϕ, according to which one must eliminate β with the aid of equation (1). These extremely complicated formulae have, however, never been carried out.

SUBBRANCH A: TRANSFORMED; ORDER b: AITOFF-ized (AITOFFIERT)

(SNo. 185, 186)

This order contains transformed, double-symmetrical projections, their process of transformation going back to Aitoff. Aitoff began with the polar grid of the hemisphere in the transverse-axical radial projection (Mercator, SNo. 24), left the x coordinate for each point (ϕ / λ) unchanged, but doubled the y coordinate and gave the thus obtained point the designation (ϕ /λ), so that a reproduction of the full global grid was produced. In somewhat generalized form, this process, which one might call *Aitoff-izing* might be defined as follows: From the hemisphere-grid of a doubly-symmetrical original projection I with cooridinates $x(\phi /\lambda)$ and $y(\phi /\lambda)$, one obtains, by means of Aitoff-izing, a doubly-symmetrical projection II with the coordinates $X(\phi/2\lambda) = mx(\phi /\lambda)$ and $Y(\phi/2\lambda) = 2ny(\phi / \lambda)$. Thus, Aitoff-izing is an expanding (as in the previous order a), but with the following refiguring of the meridians. If the original projection had a shape-true point $\phi = \lambda = 0$, then this point remains shape-true even after the Aitoff-izing, if the ratio of increase $\frac{2n}{m}$ on the equator and central-meridian is equal to the refiguring number 2, that is, $m = n$. Otherwise two shape-true points as with the expanding are produced, rather than the one, in which case the equator is increased n times and the central-meridian m times.

SNo. 185 renders *"Aitoff's Planisphere"*. Aitoff produced it from the transverse-axical projection (SNo. 24), which employed m = n = 1. The equations

$$\cos\sqrt{x^2 + y^2} = \cos\lambda\cos\phi;\quad y = x\sin\lambda\cot\phi$$

apply for the starting projection; from these equations result the equations for No. 185:

$$X(\phi/2\lambda) = x(\phi/\lambda);\quad Y(\phi/2\lambda) = 2y(\phi/\lambda)$$

When this transformation takes place, the straight-line characteristic of the great circles through the point $\phi = \lambda = 0$, as well as the general central-equal distance, are lost. But true-scale on the equator and central-meridian and true-shape at point $\phi = \lambda = 0$ are maintained. The world map border is an ellipse with an axis length π and 2π .

Hammer pointed out [79] that area-similarity is maintained when Aitoff-izing an area-true map. As far as the uniform-globe is concerned, true-area is kept, if 2mn = 1 in the transformation equations. Hammer himself obtains the Hammer Planisphere (SNo. 186) by Aitoff-izing the area-true radial transverse-axical projection by Lambert (SNo. 23) with the values $m = n = \frac{1}{\sqrt{2}}$. The equations that apply for the original projections are $x^2 + y^2 = 4\sin^2\frac{\delta}{2}$; $y = x\sin\lambda\cot\phi$ where $\cos\delta = \cos\phi\cos\lambda$, thus $x^2 + y^2 = 2(1 - \cos\phi\cos\lambda)$ From this, the result for No. 186: $X(\phi/2\lambda) = x(\phi/\lambda):\sqrt{2}$ and $Y(\phi/2\lambda) = \sqrt{2}\,y(\phi/\lambda)$.

True-area in the whole map and true-shape at point $\phi = \lambda = 0$ are maintained. The world map border becomes an ellipse with the semi-axes $\sqrt{2}$ and $2\sqrt{2}$. In the paper of the American Coast and Geodetic Survey [80], the Hammer Planisphere is incorrectly attributed to Aitoff.

There is little use in Aitoff-izing other doubly-symmetrical projections, if one does not want to attain more favorable shape relationships on an area-true projection, for example, by means of $m \gtrless n$ and $2mn = 1$, similarly to how SNo. 183 was produced from SNo. 23. A cylinder-circular projection, Aitoff-ized, becomes itself, if y is in equal ratio to λ for ϕ = constant, the projection thus being secondary-circle-spaced. This applies in family II for the branch A. Thus, only the non-secondary-circle-spaced projections, such as SNo. 153-157 by Apianus, Glareanus and van der Grinten, would yield new forms through Aitoff-ization. One could Aitoff-ize the globular projection by Nicolosi (SNo. 158), for example, and it would be easy to draw. Its circle-shaped H would become ellipses; its central-H and basic-line would remain scale-true. But the world map would be less advantageous than the full globular projection No. 159 because of the shrinking together of the angle-space at the pole to 180°. It would be the same matter, if one were to Aitoff-ize the all-conical area-true projection No. 179, the expanded transformation (No. 184) of which appears essentially more appropriate.

SUBBRANCH A: TRANSFORMED; ORDER c: WITH OTHER MERIDIAN SPACING (SNo. 187, 188)

In the last order of the subbranch, the meridian curves remain the same as in the original projection; only their distribution is changed. It is simplest to make the meridians equal-spaced. Two examples of this class should suffice. SNo. 187, the *all-globular* projection, has the circle-shaped H in common with the full-globular No. 159, but spaces them equal-shaped. This projection was proposed by the author of this paper[65] in 1922 and has been found useful for world maps[81]. It has scale-true equator and central-meridian and circle-shaped equal-spaced meridians. The arc of the meridian λ has the radius $\rho = (\pi^2 + 4\lambda^2) : (8\lambda)$, the center-point $x = 0;\ y = \rho - \lambda$, and the regularly spaceable angle-opening y from pole to equator, where $\sin z = \pi : (2\rho)$. The coordinates of the picture-point for the global point (ϕ λ) are thus

$$x = \rho \sin \frac{2\phi z}{\pi} \;;\quad y = \lambda + 2\rho \left(\sin \frac{\phi z}{\pi}\right)^2$$

The following table gives the values of ρ and z for intervals of 15° from λ. Table: Coordinates ρ and z of the All-globular Projection (SNo. 187): [See page 127a following.]

The projection, SNo. 188, by Schmidt (1803) shows the same meridian curves as the area-true projection by Molleweide (No. 135), both of which correspond to the ellipse equations

$$\frac{x^2}{2} + \frac{\pi^2 y^2}{8\lambda^2} = 1$$

where there is true-area with the uniform globe. In Schmidt's projection, the ellipse arcs may be spaced regularly, proportional to lattitude ϕ. If one introduces a running variable ϕ and employs $u^2 = 2\cos^2\psi + \frac{8\lambda^2}{\pi^2}\sin^2\psi$ the arc length Q of the quarter ellipse from equator to pole for the meridian λ is

$$Q = \int_0^{\pi 2} u \, d\psi$$

TEXT NOT REPRODUCIBLE

Koordinaten ϱ und z des allglobularen Entwurfs (SNr. 187)

λ	15°	30°	45°	60°	75°	90°
ϱ	4,4171	2,[illegible]	1,9635	1,7017	1,5970	1,[illegible]
z	15° 55,5'	36° 52,2'	55° 7,8'	67° 23,6'	79° 36,7'	90° 0,0'
λ	105°	120°	135°	150°	165°	180°
ϱ	1,5[illegible]	1,[illegible]	1,7017	1,[illegible]	1,5483	1,9635
z	95° 47,8'	106° 15,6'	112° 37,9'	115° 17,5'	124° 45,6'	125° 52,4'

The number value for Q can be found from the Legendre Tables of elliptical integrals. The arc length $\sigma_1 = 2\phi_1\ Q:\pi$ corresponds, then, with the latitude ϕ_1, on the Schmidt map at the same meridian from the equator to the picture point $(\lambda\ \ \phi_1)$. One determines according to the integral tables, through which upper integral limit ϕ_1, the value $\sigma_1\ \ \int_0^{\psi_1}\ \psi$ is to be attained. Then the coordinates of the picture point being sought are $x = \sqrt{2}\sin\phi_1$; $y = \frac{2\lambda\sqrt{2}}{\pi}\cos\phi_1$

The true-area of the Mollweide Projection is lost in this transformation; only cell-equality is maintained and all meridians are equal-spaced. True-shape exists at the point $\phi = \lambda = 0$. In Tissot-Hammer's[82] description of this projection, the central-meridian and equator are accepted as scale-true, thus rejecting area-equality with the total surface of the uniform globe. Then only cell-equality exists. All border-meridians of the map are considered there $\lambda = \pm 90°$.

SUBBRANCH B: SHAPETRUE; ORDER a: WITH ALGEBRAIC GRID LINES (SNo. 189-192)

A second subbranch of the double-symmetrical, non-row circular projections consists of the shape-true projections. True-shape is generally determined by means of differential equations, as they have been derived in Tissot-Hammer.[83] In such projections, the N are hosts of curves according to the equation $F(x,y,p) = 0$, where p is a parameter dependent on ϕ; and the meridian H are the accompanying host of equal-rectangles (orthogonal trajectories) according to the equation $F_1(s,y,q) = 0$, where q is now a parameter dependent on λ. The functions F and F_1 are not supposed to be circle-equations in this subbranch, but fulfill the stipulations for being doubly symmetrical; that is, if p_+ and p_- correspond with opposite equal ϕ-values and q_+ and q_- correspond with opposite equal λ-values, then $F(x,y,p_+) = F(-x,y,p_-)$ and $F\ (x,y,q_+) = F\ (x,-y,q_-)$ must follow. For shape-true projections it is often appropriate to use the Mercator function

$$\varepsilon = \log \text{nat tg} \left(\frac{\pi}{4} + \frac{\phi}{2}\right)$$

with its differential

$$d\varepsilon = \sec\phi\ d\phi$$

for the designation of the global point (ϕ/λ) rather than ϕ, in other words, the x-coordinates of the corresponding point of the Mercator map.

From the equations for F and F one obtains x and y as functions of p and q and can form the functions:

$$U = \sqrt{\left(\frac{\delta x}{\delta p}\right)^2 + \left(\frac{\delta y}{\delta p}\right)^2}\ ; \qquad V = \sqrt{\left(\frac{\delta x}{\delta q}\right)^2 + \left(\frac{\delta y}{\delta q}\right)^2}$$

(which must be employed in Tissot-Hammer, p. 202, line 10, in place of the unsatisfactory equations

$$U = \sqrt{\left(\frac{\delta x}{\delta \xi}\right)^2 + \left(\frac{\delta y}{\delta \xi}\right)^2}\ ; \qquad V = \sqrt{\left(\frac{\delta x}{\delta \lambda}\right)^2 + \left(\frac{\delta y}{\delta \lambda}\right)^2}\ !)$$

The stipulation for true-shape, which must accompany the already fulfilled stipulation for rectangularity in the equations for F and F , reads then:

$$U\frac{dp}{d\xi} = V\frac{dq}{d\lambda}$$

After dividing away a somewhat equal available factor z depending on p *and* q on both sides, this equation can be stated as

$$P\,\frac{dp}{d\xi} = Q\,\frac{dq}{d\lambda}$$

where P depends only on p and Q depends only on q. Now the sought-after dependence between p and ξ on the one hand and between q and λ on the other hand results by means of integrating the two differential equations

$$P\,\frac{dp}{d\xi} = c \quad \text{and} \quad Q\,\frac{dq}{d\lambda} = c$$

where c must be employed both times as equal to the same fixed value.

In the two projections SNo. 189 and 190, the N and H are curves of the second order. In No. 189 the N are focal-point-equal (confocal) ellipses according to the equation

$$F: \frac{y^2}{p^2} + \frac{x^2}{p^2 - 1} = 1$$

and the H as rectangularly-equal to the N are the focal-point-equal hyperbolae according to the equation

$$F_1: \frac{y^2}{q^2} - \frac{x^2}{1 - q^2} = 1$$

the focal points of which are: $x = 0$; $y = p = q = 1$. For x and y as functions of the parameters p and q the equations F and F yield $x = \sqrt{p^2 - 1}\ \sqrt{q^2 - 1}$; $y = pq$
One thus finds:

$$U = \sqrt{\frac{p^2 - q^2}{p^2 - 1}}; \qquad V = \sqrt{\frac{p^2 - q^2}{1 - q^2}} \qquad \text{and} \quad Z = \sqrt{p^2 - q^2}$$

The differential equations

$$\frac{1}{\sqrt{p^2 - 1}}\ \frac{dp}{d\xi} = c \text{ and } \frac{1}{\sqrt{1 - q^2}}\ \frac{dq}{d\lambda} = c$$

are to be integrated for the determination of the dependence of the parameter p on ξ and of the parameter q on λ . One obtains the projection SNo. 189 with the fixed value $c = 1$. $q = \sin\lambda$ results in

$$\int \frac{dq}{\sqrt{1 - q^2}} = \int d\lambda$$

When $d\varphi$ is employed for $d\xi = \sec\phi\ d\phi$

the other equation reads:

$$\frac{dp}{p - 1} = \frac{d}{\cos}$$

which is identically fulfilled for $p = \sec\phi$; $\sqrt{p^2 - 1} = \mathrm{tg}\,\phi$

and $dp = \frac{\sin\phi}{\cos^2\phi}\,d\phi$

Thus, the determining equation for x and y results for the projection: $x = \sqrt{p^2 - 1}\,\sqrt{q^2 - 1} = \mathrm{tg}\,\phi \cos\lambda$ $y = pq = \sec\phi \sin\lambda$

Maurer[84] has presented another derivation of this projection according to its important characteristic which gives it great importance for navigation. All straight lines of the map surface are pictures of azimuth-equals of the globe, that is, each is a curve that has a determined fixed direction-point lying out from all its points on the same azimuth. Since there is a complex of ∞^3 azimuth-equals on the globe, a multiple by three, the bundle of all the only ∞^2 straight lines of the plane can only reproduce a multiple by two, ∞^2 azimuth-equals; and these are the ∞^2 azimuth-equals whose direction-points lie on the central-meridian $\lambda = y = 0$ reproduced as a straight line. Since the map is shape-true, the invariable azimuth of each azimuth-equal can also be read off as its angle of intersection with central-meridians, for which reason the seamen use this map as the so-called Weir Azimuth Diagram. Naturally, one must read the invariable azimuth angle at the central-meridian on the same side as with all other meridians, and not on the opposite side, as

A. Wedemeyer[85] did in an incorrect understanding of the term "counter-azimuthal" coined by Hammer. Hammer[41] had used the term *counter-azimuthal* in referring to projections which reproduce the azimuth-equals of only one direction-point as straight lines intersecting themselves at its picture-point at the proper angles; Hammer did this after Craig had referred to such a map which was not shape-true in the rest of the map space and had Mecca as direction-point (cf. SNo. 200). According to this definition, projection No. 189 is also counter-azimuthal, since it fulfills the stipulation for infinitely many direction-points, and since the fulfilling of this stipulation at only one direction-point already makes it counter-azimuthal. It is the *shape-true azimuth-equal* map, in which all straight lines are azimuth equals. Its equations are found mentioned in Littrow[86], but he asserted incorrectly that its meridians and circles of latitude were hyperbolae. Maurer published the first more exact examination of this map in 1905[84] and 1911[70] and included such maps. The world map (Illustration 13, plate VI, a and b) fills out the surface two times as the Riemann Double Surface with the equator and its length increases, the meridians $\lambda = \pm 90°$ as branching-intersections. The picture of both earth poles is the infinitely distant straight line of the plane. Whereas the polar grid consists of hosts of confocal ellipses and hyperbolae, the primary- and secondary-circles of the pair of points $\phi = 0$ and $\lambda = \pm 90°$ produce and Lambert-type circle-grid with the constant $n = 2$, the context of which has already been presented by Wedemeyer[87] in 1918. It would not be appropriate, however, to place the azimuth-equal map into the system as a Lambert circle-grid, since the concept of the azimuth is linked to the meridians and the straight lines of the map surface are azimuth-equals only when the meridians are hyperbolae. A polar grid which is a Lambert circle-grid is not an azimuth-equal map. For this reason the designation *Littrow-Maurer Azimuth-equal Map* for projection SNo. 189 seems justified.

If, in the grid No. 189, one considers the ellipses as meridians and the hyperbolae as circles of latitude, then one has the projection (SNo. 190) by Firoini[83]. Its equations, then, are: $x = \sec \lambda \sin \phi$; $y = \operatorname{tg} \lambda \cos \phi$

The map consists of two full planes, the center points of which are the points ($\phi = 0$; $\lambda = 0$) and ($\phi = 0$; $\lambda = 180°$) The equator is an infinitely long double straight line; and the infinitely distant straight line is the picture of the meridian $\lambda = \pm 90°$. The poles are reproduced as straight lines, $y = 0$; $x > 1$ and $y = 0$; $x < -1$, while the central-meridian is pressed together on the line segment $y = 0$; $-1 < x < 1$. This shape-true projection is not an azimuth-equal map.

The H and N of the projections SNo. 191 and 192 are curves of a higher order. No. 191 is a transformation of No. 189 and is presented because the poles in No. 189 fall infinitely distant and the shape-true azimuth-equal map is thus not usable in the polar areas. Its transformation by means of reciprocal rays brings the pole into the center of the map and maintains true-shape, while the straight azimuth-equals of No. 189 become likewise well usable circles.

The equations for the coordinates x,y of the projections SNo. 191 are formed from those for No. 189, $x = \operatorname{tg}\phi\cos\lambda$; $y = \sec\phi\sin\lambda$ through the transformation equations

$$\frac{y_1}{x_1} = \frac{y}{x} = \frac{\operatorname{tg}\lambda}{\sin\phi}; \quad y_1^{\,2} + x_1^{\,2} = \frac{1}{x^2 + y^2} = \frac{1}{\operatorname{tg}^2\phi + \sin^2\lambda}$$

and read $$x_1 = \frac{\cos\lambda\sin\phi\cos\phi}{1 - \cos^2\lambda\cos^2\phi}; \quad y_1 = \frac{\sin\lambda\cos\phi}{1 - \cos^2\lambda\cos^2\phi}$$

The equation for the $H(\lambda)$ and/or $N(\phi)$ is obtained when one eliminates from the equation

$$\frac{y_1}{x_1} = \frac{\operatorname{tg}\lambda}{\sin\phi}; \quad y_1^{\,2} + x_1^{\,2} = \frac{1}{\operatorname{tg}^2\phi + \sin^2\lambda}$$

ϕ the first time and λ the second time; the result, as equations of the fourth order (Lemniskaten): $\sin^2\lambda\,(x_1^{\,2} + y_1^{\,2})^2 = y_1^{\,2} - x_1^{\,2}\operatorname{tg}^2\lambda$; $\operatorname{tg}^2\phi\,(x_1^{\,2} + y_1^{\,2})^2 = x_1^{\,2} + y_1^{\,2}\sin^2\phi$

The symmetry straight lines of the projection are $y_1 = 0$, i.e., the meridians $\lambda = 0$ and $\lambda = \pi$ on the one hand and $x_1 = 0$, i.e., the meridians $\lambda = -\frac{\pi}{2}$ and $\lambda = +\frac{\pi}{2}$ within the line segment $-1 < y < +1$ and the equator on the straight line $x_1 = 0$ outside of that line segment. The shape-true circle-grid which applied in the Littrow-Maurer map for the primary- and secondary-circles of the pair of points $\phi = 0$; $\lambda = \pm\frac{\pi}{2}$, i.e., $x = 0$; $y = \pm 1$, remains a shape-true circle-grid for the same pair of points in the new map also, since this point pair corresponds to itself in the transformation through reciprocal rays. But even here, the property which prompted the including of SNo. 191 here, namely that the azimuth-equals with direction-points on the central-meridian are circles, would not be fulfilled if that circle-grid were the polar grid of the map. A map-grid of this projection with meridians and circles of latitude from degree to degree was calculated and sent along in 1931 with the "Graf Zeppelin" for its arctic trip as a radio positioning grid; it was published in 1933[89]. The same Lemniskaten grid (from 15° to 15°),represented by the dotted lines in figure 1, is a reproduction of the already mentioned work by von Wedemeyer[87] in 1918, if one considers the symmetry lines running from left to right as meridian $y = 0$ and the even numbers on it as polar intervals. The other symmetry line on the inner line segment means the meridians $\lambda = \pm 90^{\circ}$ with polar intervals to the equator noted; in the outer parts it means the equator itself, which is the branching-intersection of the Riemann Double Plane and on which filling out the noted numbers to 90° would yield the even numbers of the meridians λ.

Following is projection SNo. 192, proposed by F. August[90] in 1874. Its H and N are evolvents of epicycloids. It renders one of the best possible shape-true world maps, which lies completely within the finite and gives every point of the globe without exception as a shape-true point, even in appearance. The world map is derived as shape-true reproduction of the polar grid of a transverse-axical all-circular projection (SNo. 170) with a central-H-length = 2 in such a manner that the infinite plane of the original map is reproduced in the interior of an epicycloid. This epicycloid describes a point of a circle of radius 1/2 if the circle rolls off on another one of radius 1. The latter circle in the all-circular map consists of the meridian $\lambda = \pm\frac{\pi}{2}$ and the poles are reproduced in themselves as the points $y = 0$ and $x = \pm 1$. If one calls Θ the angles between the central-meridian and the line joining the fixed and the rolling circle, then these equations apply for the epicycloid named, which gives the meridian $\lambda = \pm\pi$ as world map border:

$$x_1 = \frac{3}{2}\cos\Theta - \frac{1}{2}\cos 3\Theta$$

$$y_1 = \frac{3}{2}\sin\Theta - \frac{1}{2}\sin 3\Theta$$

or, in the form for the complex number levels,

$$u_1 = x_1 + iy_1 = \frac{3}{2}e^{i\Theta} - \frac{1}{2}e^{3i\Theta}$$

The equations for the all-circular map to be produced are:

$$z = x + iy; \qquad x = \frac{\sin\phi}{1 + \cos\lambda\cos\phi}$$

$$y = \frac{\sin\lambda\cos\phi}{1 + \cos\lambda\cos\phi}$$

On this map, $x = 0$; $y = \mathrm{tg}\,\frac{\pi}{2}$ applies for the equator; $y = 0$ applies for the meridian $\lambda = 0$; $x = \mathrm{tg}\,\frac{\phi}{2}$; and for the meridian $\lambda = \pm\pi$, $y = 0$, $x = \pm\cot\frac{\phi}{2}$.

For the transformation of the August Projection, one must form $\frac{1 \pm \sqrt{1 - z^2}}{z}$, where $z = x + iy$. Then this complex equation applies for the transformed projection: $u = x_1 + iy_1 = \frac{p}{2}(3 - p^2)$. One finds for its equator $x_1 = 0$; $y_1 = \frac{1}{2}\left[tg\,\frac{\lambda}{4}\right]\left[3 + tg^2\,\frac{\lambda}{4}\right]$; for its meridian $\lambda = 0$ $y_1 = 0$; $x_1 = \frac{p}{2}(3 - p^2)$, where $p = \left[\cos\frac{\phi}{2} - \sqrt{\cos\phi}\right] : \sin\frac{\phi}{2}$; and for its border meridians $\lambda = \pm\pi$, $x_1 = \frac{3}{2}\cos 3\Theta - \frac{1}{2}\cos 3\Theta$; $y_1 = \frac{3}{2}\sin\Theta - \frac{1}{2}\sin 3\Theta$ where $\cos\Theta = tg\,\frac{\phi}{2}$. August referred to a relatively simple geometric construction for an arbitrary point (ϕ/λ) of his map.

The following comparison shows how fortunately this projection holds its own as a world map among the shape-true world maps which have the whole earth represented in a finite picture. If one assumes the length distortion in the center of the map $\phi = \lambda = 0$ to be equal to 1, then the August map suffers the greatest length distortion on the border meridian $\lambda = \pm\pi$, where it reaches the amount 4 for $\phi = 0$ and increases with increasing ϕ up to the amount 8 for $\phi = \pm\frac{\pi}{2}$. Of the shape-true circle-grids, however, those with the constant $n \geqq 1$ expand into infinity; and for those remaining finite, with constant $n < 1$, the length distortion becomes $\frac{n}{2}\sec^2\frac{n\pi}{2}$ at $\phi = 0$ on the meridian $\lambda = \pm\pi$ and increases to ∞ for $\frac{\pi}{2}$

SUBBRANCH B: SHAPE-TRUE; ORDER b: WITH NON-ALGEBRAIC GRID-LINES

(SNo. 193, 194, 218)

The projection by Eisenlohr[91] (1870), SNo. 193, is a bit better than the August Projection, as far as length distortion is concerned, and August himself mentions this. The length distortion in the Eisenlohr Projection has its highest value 5.83, everywhere the same on the entire world border meridian $\lambda = \pm\pi$, as compared with the value 1 at the point $\phi = \lambda = 0$. The equations for this projection read, in our complex way of designation:

$$y + ix = \frac{2}{i}\,(v - iu) + 2\sqrt{2}\left[e^{v-iu} - e^{-(v-iu)}\right] \quad \text{where}$$

$$\operatorname{tg} u = \sin\frac{\phi}{2} : \left(\cos\frac{\phi}{2} + \cos\frac{\lambda}{2}\sqrt{2\cos\phi}\right) ;$$

$$v = \frac{1}{2}\left[\frac{\cos\frac{\phi}{2} + \sqrt{\cos\phi}\,\cos\left(4 - 2\right)}{\cos\frac{\phi}{2} + \sqrt{\cos\phi}\,\cos\left(4 + 2\right)}\right]$$

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They are, however, so complicated, that they have never yet really been used for a practical production of a world map. Until further findings, the August Projection must be designated as the shape-true world map with the least area- and length-distortion. But it has, as Tissot-Hammer[92] state, not received the attention deserved. The work by Eisenlohr is not in Volume 71, as stated by August, but in Volume 72 of the Crell Journal.

Two further double-symmetrical, shape-true projections have been proposed independent of one another by C.S. Pierce (1879), SNo. 218, and by M.E. Guyou (1887), SNo. 194. They are simply related in that, when one map contains one of these projections as polar-grid, the other projection represents the grid of primary- and secondary-circles of the intersection-point of central-meridian and equator. In other words, the Guyou Grid is the polar grid in transverse-axical inclination when the Peirce Grid applies as polar-grid in earth-axical inclination, and the other way around. This relationship does not seem to have been recognized before. It seems especially peculiar that the projection by the American Peirce[94] is not mentioned in the text book for map study of the *American Coast and Geodetic Survey* , but only on the closely related, just otherwise inclined, projection of the Frenchman Guyou[95] which is eight years younger. Guyou's projection belongs with the double-symmetrical projections of our family III, whereas that of Peirce shows different relationships of symmetry and therefore belongs in family IV. First, however, we want to treat the so-called *quincunctial projection* (five-form projection) by Peirce (SNo. 218), since it was the first to be proposed and can be somewhat more easily described.

If one designates the picture of the global point (δ/λ) on the complex map surface with $(x + iy)$, then Peirce employs cosam $(x + iy + K;k) = \mathrm{tg}\,\frac{\delta}{2}\,(\cos\lambda + i\sin\lambda)$, where $k = \frac{1}{\sqrt{2}}$, and the angle of his modulus $= \frac{\pi}{4}$ and K is the elliptical integral of the first class,

$$\int_0^1 \frac{dz}{\sqrt{(1-z^2)\left(1-\frac{z^2}{2}\right)}}.$$

The map picture, the classification of which is indicated in Illustration 14 (plate I), consists of infinitely many congruent squares adjacent to each other on all sides, the periphery of which, such as $A_1\ A_2\ A_3\ A_4$, is the picture of the equator, whereas square-centers alternately in both directions of the adjacency of the squares indicate the North Pole and the South Pole. Every four neighboring poles of the same designation (as $S_1\ S_2\ S_3\ S_4$ and/or $N_1\ N_2\ N_3\ N_4$), which are in the corners of a greater world map square, form a five-form--a quincunx--with the opposite pole (N_1 and/or S_1) in their center; thus the name of the projection. One may also call such a world map square a four-pointed star, the points of which make up the four-fold picture of the same global point. Other star-projections are the combination projections SNo. 228-231. Every semi-meridian $\pi = n\frac{\pi}{4}$ (n = whole number) between pole and equator on our map is a straight line; and these straight lines intersect at the pole at the proper angles. At the equator corners A, however, the apparent angle discrepancy increases up to 90°.

The advantage of this projection is that it does not have a prescribed world map border. Full world maps in the above described star form are naturally only squares, the four corners of which are pictures of the same earth pole. On the other hand, that quadrilateral is also a full world map, whose straight parallel laterals, such as BE or CD, represent the half equator and are separated from each other by one fourth of a length of the equator ($=A_1\ A_3$), whereas the two other laterals can be arbitrary curves, BC and EB, which must only be congruent and parallel. The arrangement of the globe eighths lined up beside each other can be seen from the reproduction; each of the globe eighths appears as an iscoceles-rectangular triangle. The four parts of the northern hemisphere are disignated with 1,2,3,4, and those of the southern hemisphere with I, II, III, IV. The dotted curves correspond approximately to the meridians $\lambda = 15^{\circ}$ and $\lambda = 195^{\circ}$. While the squares [S_1 S_2 S_3 S_4], [N_1 N_2 N_3 N_4], [N_1 N_4 N_5 N_6], [N_1 N_6 N_7 N_8], [N_1 N_8 N_9 N_2] are full world maps, the equally great squares with corner points A are not. On the other hand, not only rectangles, such as [A_9 $A_{10}$$A_2$ A_1], [A_3 A_8 A_5 A_1], [A_4 A_2 A_6 A_{11}] and [A_3 A_2 $A_{11}$$A_{12}$] are full world maps, but also the already mentioned quadrilateral B C D E. Although overall true-shape is proven in Peirce's projection, when subjected, even in the corners of the equator square, to microscopic mathematical examination, a world picture with square equator really appears macroscopically quite unsatisfactory, no matter if a hemisphere is torn up into four three-cornered lobes in the star-form such as S_1 S_2 S_3 S_4, or if both hemispheres cohere at only one quarter of the equator in the rectangle such as A_3 A_5 A_6 A_4.

The following must be noted with respect to the family to which Peirce's grid should belong: It is not possible to give such a closed world map such a delimitation that the grid shows the double-symmetry characteristic of family III, since the equator does not form a uniform straight line as symmetry-line of the grid, even though it is symmetrical to each of its four straight-line parts. We must assign the projection to family IV. This is another matter if we consider the straight-line full meridian $S_1 N_1 S_3$ in Peirce's map as the great secondary-circle and go over to the grid of primary-and secondary-lines, i.e., to Guyou's projection. This grid is doubly-symmetrical and belongs to family III. It, too, is a quincuncial grid which fills out the entire map surface with squares, each representing a hemisphere, but now an east- or west-hemisphere, rather than a north- or south-hemisphere. The particular world map is again either a rectangle with lateral ratio of 1:2, as FGHJ in Illustration 14 (plate I), in which $A_2 A_3$ represent two-fold the north pole $A_1 A_2$ and two-fold the south pole and the two hemispheres connect only at one piece of meridian of 90° arc length; or a square star with a square east- or west-hemisphere in the center and another hemisphere slit up into four triangular lobes. Illustration 15 (plate V) shows a part of such a world map with 270° longitudinal difference on the straight-line equator from the north pole--reproduced as two points N--to 30° south. The ±90° longitude passes from the equator to 45° latitude as a perpendicular at the equator and is then split into two parts parallel to the equator. At the slit-points, the circle of latitude 45° appears rectangularly broken. The four quarters of the globe between meridians are designated as I, II, III, IV, whereby the arrangement of contiguous world maps is made visible. The dotted line shows how some lines in such a world map assume unnatural forms--the picture of the globe-circle of 60 arc degrees about the central point M ($\phi = \lambda = 30°$). Nevertheless, a world map according to this projection looks quite acceptable, when one can lay the rectangular meridian break in the middle of the ocean, as is done on the map in the American textbook[92]).

The mathematical derivation given by Guyou for his projection is very singular. For position-finding on the globe, he uses two hosts of "globe ellipses" which intersect themselves rectangularly. Such an elliptical line of latitude is the entirety of all points whose arc intervals on the north- (south-) hemisphere yield a constant sum from the two fixed points $\lambda = \pm 90°$; $\phi = 45°$ north (south). The curves rectangular to this host of elliptical lines of latitude are the "elliptical meridians". Each one of these connects all points whose arc intervals on the east- (west-) hemisphere yield constant sums from the two fixed points $\phi = \pm 45°$ and $\lambda = 90°$ east (west). That latitude ϕ applies as elliptical latitude coordinate ϕ_0 of a latitude ellipse at which this globe ellipse intersects the zero meridian $\lambda = 0$. That longitude λ, at which the concerned globe ellipse intersects the equator, applies as elliptical longitude coordinate λ_0 of a meridian ellipse. The equations which apply between these different coordinates on the globe are:

$$\sin\phi = \sin\phi_0 \sqrt{1 - \frac{1}{2}\sin^2\lambda_0}; \quad \cos\phi \sin\lambda = \sin\lambda_0 \sqrt{1 - \frac{1}{2}\sin^2\phi_0}$$

Guyou reproduces the grid of elliptical latitude- and longitude-lines of the globe now as grid of straight lines x = const and y = const, shape-true on the map surface, the equator x = 0 and the central-meridian y = 0, according to the reproduction-equations:

$$x = \int^{\phi_e} \frac{d\phi_e}{\sqrt{1 - \frac{1}{2}\sin^2\phi_e}} \; ; \qquad y = \int_0^{\lambda_e} \frac{d\lambda_e}{\sqrt{1 - \frac{1}{2}\sin^2\lambda_e}}$$

(Guyou uses other letters in his paper

Our designations:	x	y	ϕ	λ	ϕe	λe
Guyou's designations:	y	x	L	G	λ	γ

)

The shape-true projections of Peirce and Guyou, with *quadratic* hemisphere grid are nevertheless the most advantageous special cases of the more general of such shape-true, double-periodical projections with *rectangular* grid, where each rectangle represents a hemisphere. The equations of Guyou may be generalized in the forms:

$$\sin\phi = \sin\phi_e \sqrt{1 - \cos^2 F \sin^2\lambda_e}$$

$$\cos\phi \sin\lambda = \sin\lambda_e \sqrt{1 - \sin^2 F \sin^2\phi_e}$$

$$x = K \int_0^{\phi_e} \frac{d\phi_e}{\sqrt{1 - \sin^2 F \sin^2\phi_e}} \; ;$$

$$y = K \int_0^{\lambda_e} \frac{d\lambda_e}{\sqrt{1 - \cos^2 F \sin^2\lambda_e}}$$

where K is a fixed scale value and the focal points of the groups of globe ellipses are $\lambda = \pm \frac{\pi}{2}$ $\phi = \pm F$. Guyou, who pointed this out, chose $K = 1$, $F = \frac{\pi}{4}$ for his projection.

FAMILY IV: LESS REGULAR; NEITHER TRUE-CIRCULAR NOR DOUBLE-SYMMETRICAL

(No. 197-225)

The projections of the last subbranch in family III, in which the H are parallel straight lines, will be mentioned later in connection with family IV, to which we now turn our attention. In connection with SNo. 92, we have already mentioned SNo. 197 and 198 of this family, the cylinder-perspective projections which are no longer double-symmetrical if the sighting point leaves the center of the globe; also mentioned in the note on page 67 were the cylinder-circular projections SNo. 208-214, which are produced when the determining equations, really valid only for a hemisphere with respect to projections No. 104-106, 113-115, 124-126, 132, 134 are applied for an entire world map. We have also already mentioned the cylinder-radial interval- or area-true projections of family IV, which are produced from the double-symmetrical cylindrical projections SNo. 87-91 when their N-lines are substituted for with parallel curves. No. 201-204 are such projections, in which the parallel curves are circle-arcs with radius r. No. 201, the interval-true projection on a contiguous-cylinder, has the equations $y = \lambda$; $x = \phi + r - \sqrt{r^2 - \lambda^2}$ The entire arc length of each N is $L = 2r\alpha$, where $\sin \alpha = \pi : r$. Here L becomes always greater than the basic-circle 2π on the globe, since $r > \pi$. True-shape exists then only at point $\phi = \lambda = 0$.

The equations for SNo. 202, the interval-true projection on an intersecting-cylinder, are:

$$y = n\lambda; \quad x = \phi + r - \sqrt{r^2 - n^2\lambda^2}; \quad n < 1; \ r > n\pi$$

Now the entire length of each N becomes $L = 2r\alpha$, where $\sin\alpha = n\pi : r$

$2r\alpha$ can then become equal to the entire length of any two $N \pm\phi_m$ on the globe, and true-shape is produced at the two points $\lambda = 0$ and $\phi = \pm\phi_m$. Example:

$n = 1 : \sqrt{2}$, $r = \pi$, $\alpha = 45°$, $\cos\phi_m = \pi:4$; $\phi_m = \pm 38.2°$.

Shape-true points, $\lambda = 0$ and $\phi = \pm\ 38.2$.

$y = \lambda;\ x = \sin\phi + r - \sqrt{r^2 - \lambda^2}$ apply for the area-true projection on a contiguous-cylinder (SNo. 203). Arc length of the N and true-shape are as for No. 201. $y = n\lambda;\ x = [\sin\phi + r + \sqrt{r^2 - \lambda^2}] : n$ apply for the area-true projection on intersecting-cylinder (No. 204). For arc length of the N the same applies as for No. 202. There is true-shape, however, only for $\lambda = 0$ and $\phi = \pm\phi_w$ where $\phi_w = n$. Example:

$n = 1:\sqrt{2}$; $r = \pi$; $\alpha = 45°$; $\phi_m = \pm 38.2°$, but $\phi_w = \pm 45°$.

If one rejects the equidistance of the cylinder-rays H, then one can make the N-circle-arcs equal-spaced. SNo. 205 and 206 are such projections. The first is interval-true, the other zone-true.

Special Stipulations for Projections

In order not to go too far afield in discovering further projections, it is expedient to refer to practical stipulations which lend the map valuable properties or facilitate, by means of the grid, the solution of geographic or navigation problems on the earth or astronomical problems in the sky. We treat this question in a general manner and state which projections fulfilling such stipulations have already been named in families I, II, and III, and which of the ones with less grid regularity should yet be assigned to family IV. The stipulations concerned here refer either to the reproduction of angles, areas or lines which are of importance on the globe for geographic or nautical (astronomical) purposes. We consider the global great-circles (orthodromes) as the shortest (geodetic) lines of the globe, the course-equals (loxodromes) as lines of constant course on the globe, and the azimuth-equals as locating-lines of those points from which a direction-point appears in constant azimuth.

A. Stipulations Concerning the Angles (SNo. 217, 220)

The stipulation for true-shape in the entire map is given by means of differential equations, to which we have referred on page 67, and which make extremely diverse grid-lines possible, so that such projections appear in all families. The only shape-true radial projections we find in family I are the all-circular (No. 2) and the conical maps (No. 39 and 40); in family

II, the Etzlaub-Mercator Maps No. 94 and 95; in family III, the row-circular shape-true circle-grids of the first class, No. 169-173, and of the second class, No. 181, as well as the non-row-circular No. 189-194 of Littrow-Maurer, Fiorini, August, Eisenlohr and Guyou. As belonging to family IV we consider the projection by Peirce (SNo. 218), which does without the double-symmetry and the shape-true circle-grids after Lambert and Lagrange, in which not the equator, but another circle of latitude is reproduced as a straight line and which are therefore not symmetrical to the central-H; plus the lateral world maps of such shape-true circle-grids, which inversely are only symmetrical to the equator. The first have been included as SNo. 217, the second as SNo. 220, in family IV.

True-shape at only two points, or at only one, is only stipulated when the great circle is a straight line through the shape-true point; more about this under C below.

B. Stipulation for True-area (SNo. 203, 204, 212, 213)

This stipulation, too, designated by the differential equation

$$\frac{\delta x}{\delta \phi}\ \frac{\delta y}{\delta \lambda} = \frac{\delta x}{\delta \lambda}\ \frac{\delta y}{\delta \phi} = \cos \phi$$

can be fulfilled in manifold forms of the grid-lines, so that area-true projections appear in all the families. In family I we find the radial point-maps by Lambert (SNo. 23), the radial ring-maps No. 28 and 29 and the conical projections No. 51

and 52 (Albers). Family II offers the true-cylindrical maps No. 87 by Lambert, No. 89 by Behrmann and the extremely numberous untrue-cylindrical SNo. 131-149, among which are ones by Collignon, Mollweide, Nell, Hammer, Wagner, Crastner, Prepetit-Foucaut and Bourdin. In family III we find the all-conical projection SNo. 179, the transformed projections No. 183, 184, 186; and even among the combination projections of family V, we shall discover true-area. We have already pointed out the area-true maps of family IV (SNo. 203, 204, 212, 213). For more concerning an area-true counter-azimuthal projection (SNo. 219), which likewise belongs in general to family IV, see E2.

C1. Straight Lines as Pictures of Great Circles of the Globe (SNo. 199, 195)

Maps in which *all straight lines* are pictures of great circles are called straight-directional or orthodromic. Such are the central-perspective projection (SNo. 1) and its collinear reproductions, also Field No. 81 on page 182. They contain in general two shape-true points (SNo. 182 in family III), whereas the special case (SNo. 1) indicates greater regularity of central-circularity with only one shape-true point. The term "gnomonic" is to be limited to only this special case.

The stipulation that the great-circles of a global point be reproduced as straight lines which intersect themselves at proper angles is called *radial-directionality* (Radstrahligkeit) (sometimes also called cluster-radiality (Buschelstrahligkeit)).

If one stipulates *radial-directionality for two points*, then the *double-radial-directional* projection is therewith fully determined, including the scale, and it is in this manner that one obtains the straight-directional projection SNo. 182.

A projection has been proposed, which is *radial-directional* at only *one point S*, whereas not radial-directionality, but cylinder-radiality is stipulated for a second point, the earth pole, that is, the meridians should be equidistant parallels. This suggestion (SNo. 199) was offered by Schoy[96]. This projection, which is *radial-* and *cylinder-radial*, has been incorrectly designated as azimuthal, which does not take into account that the central-circularity necessary for this is missing.[97] His equations read: $y = \lambda$; $x = \lambda[\cos\phi_0 \operatorname{tg}\phi - \sin\phi_0 \cos\lambda]:\sin\lambda$

It is to be noted here that one can combine the cylinder-radial reproduction of the great circle which is perpendicular to the central meridian with cylinder-radial reproduction of the meridians, thus achieving a *double-cylinder-radial* projection. This corresponds with a naive attempt to simply consider the globe as a plane. Its equations would be $y = \lambda$ $x = \psi$, where ψ is the angle which each great circle of the second group forms with the equator, thus $\cot\psi = \cot\phi \cos\lambda$ To be sure, the grid in the vicinity of the point $\phi = \lambda = 0$ is very advantageous and exhibits scale-true equator and central-meridian. But it is extremely disadvantageous for areas in the vicinity of $\lambda = \pm\frac{\pi}{2}$. Each of these two meridians shrinks together into a pair of points $y = \pm\frac{\pi}{2}$; $x = \pm\frac{\pi}{2}$ whereas the entire line segments $\frac{\pi}{2} < x < \frac{\pi}{2}$ on the lines $y = \pm\frac{\pi}{2}$ are the pictures of the points ($\phi = 0$; $\lambda = \pm\frac{\pi}{2}$) According to its symmetry relationships, this projection (SNo. 195) belongs to family III.

C2. Circles as Pictures of Globe Great Circles

Maps which reproduce all great circles of the globe as circles are designated as *great-circle-true*. For the radial one among them, the radius rule stated by Schols and already stated on page 25, applies:

$$r = 2\ \mathrm{tg}\ \delta : (1 + \sqrt{1 + k^2 \mathrm{tg}^2 \delta}) = 2 : [\cot \delta + \sqrt{k^2 + \cos^2 \delta}]$$

From among these radial great-circle-true projections, the central-perspective SNo. 1, with the constant $k = 0$, deserves special attention, since it covers the surface of the map two times, with one global great circle represented as infinitely distant straight lines of the picture plane and all great circles as straight lines. For $k > 0$, the radial-circular map covers the picture plane only once and represents the point $\delta = \pi$ as infinitely distant straight lines, whereas the circle $\delta = \frac{\pi}{2}$ obtains the radius 2:k in the picture.

The following example shows how bizarre a form a reproduction can assume, which fulfills the stipulation for being great-circle-true, a sensible stipulation per se: A straight-directional map with two shape-true points, A and B, is great-circle-true, since all great circles on it are straight lines. We reproduce this map by means of reciprocal rays from point M_1 which is the picture of the mid-point of the arc AB on the globe. In the resulting picture all the great circles are represented by circles. The point M of the globe is now represented as the inifinitely distant straight lines, and the entire global great circlè, the middle of which is the point M of the globe, is represnted now by a point O. All great circles of the globe are represented by the cluster of all circles passing through 0; and the global great circles which pass through the global point A and/or B are the circle-cluster through 0 and the picture-point A_1 of A in the picture, and/or the circle-cluster through 0 and the picture-point B_1 of B, while true-shape exists at the same time at points A_1 and B_1.

D1. Straight Lines as Pictures of Course-Equals (Loxodromes) (SNo. 207, 224, 225)

The stipulation for reproducing all course-equals as straight lines is fulfilled most usefully for navigation by the true-shape marine map according to Etzlabu-Mercator (SNo. 94 and 95). Aside from that one, every collinear reproduction of the Mercator map represents the course-equals as straight lines. True-shape is generally lost with perspective reproduction.

However, the map remains yet a cellular (zenithal) prime-point-map, if the point of vision lies infinitely distant. Then, the map also remains cylindrical, if the picture plane is parallel to the central meridian or to the equator of the map being reproduced, as with SNo. 96; on the other hand, with any other position of the picture-plane, the parallel straight-line meridians lie oblique to the equator (SNo. 224). If the point of vision lies in infinity, then the map is neither a prime-point map nor cellular, but remains cylinder-circular and secondary-circle-spaced, if the picture plane stands parallel to the equator of the map to be reproduced (SNo. 207), whereas these two properties are also lost with another position of the picture plane (SNo. 225). All these projections have no practical significance; however, Maurer was able to more clearly present properties of the course-equal projections in one grid after No. 96, about which ambiguity had existed in the literature.

D2. Circles as Pictures of Course-Equals
(SNo. 217, 220-223)

One obtains a map in which all course-equals of the globe are represented as circles, if one reproduces by means of reciprocal-rays one of the projections named under D1 with straight course-equals. Then, true-shape is maintained, if the map being transformed is a Mercator map. On a projection obtained in this manner, the north and south poles converge at one point in infinity, so that such a grid can be used for examining the angle relationships at the poles, which are not accessible on the Mercator map. As a matter of fact, proof has been given with such a grid that the Mercator map is shape-true at the poles also, although the appearance of the parallel meridians contradicts this[28]. Such a shape-true map with circle shaped course-equals is at the most simple-symmetrical, for example, with the equator as symmetry-line when the point of intersection of the reciprocal rays lay on the equator (SNo. 222), or fully unsymmetrical, when this point of intersection lay neither on the equator nor on the central-meridian (SNo. 223).

One can consider a map like SNo. 222 as a special case of one of the before mentioned lateral maps with shape-true circle-grids, which are in our system as SNo. 220, namely, as that special case where the entire center of the map shrinks together to one point. Also the general case of a lateral map with shape-true circle-grid represents an only simple symmetrical grid (SNo. 221) of family IV, if the central map is not a point and north and south poles thus lie apart from each other, and if another circle of latitude, rather than the equator, is reproduced as straight line (SNo. 217). There are also certain course-equals in the shape-true circle-grids SNo. 169-173 and 217, as in their lateral maps No. 220 and 221, namely, the meridians and circles of latitude as course-equals of the course angles 0° and 90°, which are reproduced as circles. But no other course-equal in them is a circle, since every other course-equal must pass through both poles, all circles in those maps, however, representing meridians through both poles.

E1. Straight Lines as Pictures of Azimuthequals (SNo. 200, 205, 216, 219, 196)

As has already been mentioned, not all ∞^3 azimuthequals of the globe can be represented with only ∞^2 straight lines of the flat surface. But all straight lines of the plane of the picture can certainly be pictures of azimuth-equals on an equal-azimuth map. The purpose of such a map can only be that the parallel azimuth of every azimuth-equal of the map be able to be read directly as intersection-angle of this azimuth-equal with the meridian of the direction-point. It must be stipulated that the ∞^1 direction-points of the straight ∞^2 azimuth-equals lie on a meridian represented by a straight line along which there is true-shape. The derivation for the projection (SNo. 189), which accomplishes this, was stated in 1905 by Maurer. Naturally, all straight lines of the picture surface remain pictures of the same bundle of ∞^2 azimuth-equals in every collinear reproduction of this projection. But since true-shape cannot be maintained along the meridian of the direction-point in such a reproduction, every such azimuth-equal map would miss its purpose.

Projections, which reproduce the azimuth-equals of a direction-point as straight lines intersecting at the proper angle, were designated as counter-azimuthal in 1910 by Hammer. For the sake of brevity, such a direction of a counter-azimuthal map is to be called a *counter-azimuthal point*. The projection by Littrow-Maurer (1883 and/or 1905) thus contains ∞^1 counter-azimuthal points and is chronologically the first counter-azimuthal projection. The first to have only one single counter-azimuthal point (Mecca) was offered in 1909 by Craig[99] as *retro-azimuthal*, with the added stipulation that the meridians be equidistant parallels. This is thus a *cylinder-radial counter-azimuthal map*. If the counter-azimuthal point, as in Craig's projection, does not lie on the equator, then the projection is only simply-symmetrical (family IV); it comes into family III for $\phi_0 = 0$. We include it, then, as SNo. 200 in our system. Its equations are

$$y = \lambda; \quad x = \lambda\ [\sin\phi \cos\lambda - \cos\phi \operatorname{tg}\phi_0]:\sin\lambda$$

It was shown for the center-interval-true counter-azimuthal projection by Hammer (SNo. 78) that the stipulation of a counter-azimuthal point $\lambda=0;\ \phi=\phi_0$ yields the determining equations $x = y \cot A$; $\cot A = \operatorname{tg}\phi_0 \cos\phi \operatorname{cosec}\lambda - \sin\phi\cos\lambda$, where A is the constant azimuth. $y = \lambda$ is employed by Craig as condition for cylinder-radiality, in place of the stipulation of Hammer for center-equidistance $x^2 + y^2 = \delta^2$, where

$$\cos\delta = \sin\phi \sin\phi_0 + \cos\phi\cos\phi_0\cos\lambda$$

The distribution of the circle of latitude on each meridian λ obtains the form: $x = a\cos\phi - b\sin\phi$, where $a = \lambda \operatorname{cosec}\lambda \operatorname{tg}\phi_0$ and $b = \lambda\cos\lambda$

Zoppritz-Bludau[100] include a Craig map with Berlin as direction-point.

The central-circularity of the counter-azimuthal projections SNo. 78, 79, 80 is lost with the Craig stipulation for cylinder-radiality. The result is the same if we expect true-area from a counter-azimuthal projection. In order to carry that out, one would have to reproduce the circles δ = const as such curves that the sector between such a piece of curve and each two straight azimuth-equals of the counter-azimuthal point would be equal to the piece of area that is bordered on the globe by the two azimuth-equals and the circle δ = const--a very complicated job which is hardly worth the trouble. The projection (SNo. 219) has only been included in the system in order to show that true-area and counter-azimuthality can be combined. The projection belongs in general to family IV, but it becomes double-symmetrical (family III) if the counter-azimuthal point falls on the equator.

We find counter-azimuthal maps in family I as true-circular prime-point-maps with or without equidistance (SNo. 78-80). As cylindrical ring maps, with or without equidistance, they belong to family II, branch B in the special case $\phi_0 = 0$, whereas the two forms (SNo. 215 and 216) in the general case $\phi_0 \gtrless 0$ belong to family IV.

The *double-counter-azimuthal* projection by Maurer[101] in 1919 (SNo. 196) belongs yet to the counter-azimuthal maps of family III. The two counter-azimuthal points lie on the equator $\phi = 0$; $\lambda = \pm\lambda_1$. The equations of the projection are:

$$x = 2\sin\phi\ (\cos^2\lambda_1 - \sin^2\lambda):\sin 2\lambda_1 ; y = \sin 2\lambda:\sin 2\lambda_1$$

The grid (Illustration 16, plate VII) is double-symmetrical. The meridians are parallel but not equidistant, straight lines. The circles of latitude are ellipses which all pass through the points $x = 0$; $y = \pm\lambda_1$. The poles are represented by the two circles $N_1\ N_2\ N_3\ N_4$ and $S_1\ S_2\ S_3\ S_4$. In the reproduction, the two direction-points A and B are put at $\phi=0°$, $\lambda = 50°W$ and $\phi = 0°$, $\lambda = 80°W$, that is, just about where the equator meets the coasts of South America. The cell between the meridians 20°W and 110°W lies inside the border $N_1\ N_2\ S_2\ S_1\ S_4\ N_4\ N_1$, in which the borders of the continents are also indicated with dotted lines. The adjacent global cells between the meridians 20°W to 10°E and 110°W to 140°W are formed in the two wedges N_2 A S_2 and N_4 B S_4, so that the entire meridians 10°E and/or 140°W shrink together into the points A and/or B. The rest of the global cells between the meridians 10°E and 25°E and/or between 140°W and 155°W fall in the wedges A N_3S_3 and/or B N_3S_3, by which the one hemisphere is reproduced. Furthermore, every earth point is congruent with its antipodal point. The semi-ellipse circle-of latitude-pictures cutting through the equator are not drawn, since they squeeze together too much. It is interesting to note that this map with two counter-azimuthal points lies entirely with the finite, whereas the map by Craig with only one such point, as well as the azimuth-equal map with infinitely many counter-azimuthal points, reach into infinity.

FAMILY V: COMBINATION GRIDS (SNo. 226 - 237)

Branch A: Double-symmetrical (SNo. 226,227)

Whereas a uniform mathematical expression for the coordinates x and y or for the equation for each H or for each N applied for the whole world map in the projections of families I through IV, the combination grids of family V exhibit different forms of the reproduction rule for the particular parts of the map.

In branch A are presented doubly-symmetrical projections in which the same law applies neither for the N (Branch A) nor for the H (Branch B) on the whole map. Pictures of branch A produce, as already mentioned on p.67, those doubly-symmetrical projections in which every meridian is represented as a pair of straight lines symmetrical to the equator. They are all brought together in family V under SNo. 226. The rule for the N in them does apply for the whole map.

As example of a projection in which the rule for the H applies for the whole map, while each N consists of four arcs of different radius and without break, we may use Maurer's *Isogon Map* of 1911 (SNo. 227). The central part of the map (Illustration 17, plate I) pictures the eastern hemisphere with the map center O ($\phi = 0$; $\lambda = 90^{\circ}$) and the radius OP = 1 in the transverse all-circular projection, its H-line-rule

$$x^2 + \left[y - tg\left(\frac{\pi}{4} + \frac{\lambda}{2}\right)\right]^2 = sec^2\left(\frac{\pi}{4} - \frac{\lambda}{2}\right)$$

also applying for the western hemisphere, so that all H-circle-lines can be drawn. The equations

$$(x - cosec\,\phi)^2 + y^2 = cot^2\phi = r^2{}_1$$

apply for the N-arcs inside the eastern hemisphere and its radius r_1.

If the arc $N_2N_0N_1$ of the inner hemisphere is drawn about its center M ($y = 0$; $x = cosec\,\phi$), then it is first lengthened by the two arcs N_1N_3 and N_2N_4 with the radii

$$r_2 = tg\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

the centers of which are the intersection points D_1 and/or D_2 or the polar tangents D_1OD_2 with the rays MN_1 and/or MN_2 and further about the semi-circle about center P with radius $PN_3 = PN_4 = r_3 = 2\,tg\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$.

Illustration 18 (plate IX) is a redrawing of the map of 1911, in which the directions of the isogons is from the year 1931. Aside from the strongly distorted South America, the shapes are quite good. Above all, however, such a map is much more appropriate than the usual isogon map in the Mercator projection for clarifying the general course of the isogons. Every magnetic-declination-equal passes through the two earth poles as well as through the two magnetic poles of the earth, all four of which lie here in advantageous areas of the map, whereas the Mercator map does not give the earth poles and mostly only one magnetic pole.

BRANCH B, SUBBRANCH A: TRUE-CIRCULAR (SNo. 228 -231)

In branch b, the projections are not *doubly-symmetrical*, because the equator is not represented as a straight line. In the first subbranch A we find projections, too, in which the law for the N applies for the whole map; this rule is $r = f(\delta)$ as for the true-circular projections in family I. But no uniform rule applies for the H.

We find the *star-projections* in the first order of the subbranch, in which the northern hemisphere is presented as a straight-cellular reproduction as according to family I, the southern hemisphere, however, as slit up into n prongs along certain border meridians, each of which ends up at a point representing the south pole. The three projections SNo. 228, 229, and 230 are all center-interval-true ($r = \delta$) and reproduce the H of the southern hemisphere as straight lines. The northern hemisphere of No. 228 and 229 is radial; the number of lobes of the southern hemisphere is 8 in Petermann's Star Projection, 5 in that of Berghaus. The lobes in Petermann's projection are not uniform with the border meridians 10°, 60°, 100°, 155°E and 35°, 85°, 120°, 155° W, in order to avoid distortions of Africa, Australia, America. In Berghaus' projection the lobes are congruent with the border-meridians 56°, 128°E and 16°, 88°, 160° W, so that Australia is cut through the middle. One comes out better with six uniform lobes, as is shown in SNo. 231 and 233.

SNo. 230, Steinhauser's *conoalatic* (konoalatischer) projection has a conical northern hemisphere and four lobes congruent with the border meridians 0°, $\pm 90^{\circ}$ and 180°. Since the map is projected in a sector of 240°, its north pole becomes a point, if it is projected center-interval-true on an intersecting cone through $\delta_m = 85.7^{\circ}$ ($3 \sin\delta_m = 2\delta_m$) and reproduces δ_m true to scale. The newly added star projection SNo. 231 has a northern hemisphere in area-true radial projection with the radius $r_1 = 2 \sin \frac{\delta}{2}$. For the six lobes of the southern hemisphere congruent with the border meridians 40°, 100°, 160° E, 20°, 80°, 140° W, the radius rule is the radius rule mirrored at the equator of the northern hemisphere, thus,

$$r_a = 2\left(\sqrt{2} - \cos\frac{\delta}{2}\right)$$

Then, true-area is also produced for the southern hemisphere, if we make the uniformly distributable entire length of each of its circles of latitude just as great as that of the circle of latitude in the comparable northern latitude. If one counts λ from the central meridian of the lobe represented in straight line in a lobe of the southern hemisphere $\left(\delta \geqq \frac{\pi}{2}\right)$ then these equations apply:

$$r_a = 2\left(\sqrt{2} - \cos\frac{\delta}{2}\right) \text{ and } \quad = r_i : r_a \text{ where}$$

$$r_i = 2\sin\frac{\pi - \delta}{2} = 2\cos\frac{\delta}{2}$$

Illustration 19 (plate VIII) shows this very pleasing form of the grid with slightly curved H of the southern hemisphere.

BRANCH B, SUBBRANCH B: NEITHER DOUBLY-SYMMETRICAL NOR TRUE-CIRCULAR (SNo. 232 - 237)

In Subbranch B we find no uniform law for the whole map, neither for the H nor for the N. Here, too, the first order offers *star-projections*, where, however, the equator is not a uniform curve but is an n-corner in the first class. Each side of an n-corner is symmetry-straight for the pictures of the halves of that global-cell of which a piece of the equator is the original side of an n-corner. The border meridians, as well as the central meridian of each such half of a global cell, that is, of each lobe of the southern hemisphere, are straight lines. In the subclass, each H is a pair of straight lines, and each N of a lobe is a straight line parallel to the piece of the equator of the lobe. The H can be equally or unequally spaced, the n-corners regular or not regular. In SNo. 232, we find an irregular octagon with equal-spaced H at the same border meridians as in No. 228. Jager[103] presented this projection in 1865. Illustration 20 (plate VII) shows the irregular star, the southern tips of which stand off at varying distances from the center of the star, when the central-H of the lobes are spaced scale-true, distortion of the lands of the earth nevertheless being avoided. One could have achieved the latter also with a regular six-lobed star, and thereby have had the advantage, that the equator and the six border meridians (the same as in No. 231) could be reproduced true to scale (SNo. 233). Illustration 21 (plate VIII) shows the extremely simple grid, which is so obvious that it has probably been used here and there.

It is quite simple to attain an *area-true* projection (SNo. 234) in this subclass; as example of this we chose a quincunx representation with uniform equator form and polar inclination as in the projection by Peirce (SNo. 218). The equator and each circle of latitude of the northern hemisphere becomes a square. One global triangle between the north pole and the equator-points $\phi = 0$, $\lambda = \pm\frac{\pi}{4}$ is reproduced in an isosceles-rectangular triangle with height and half hypotenuse $y = x = \sqrt{\pi}\sin\frac{\delta}{2}$, and one need only space the N-lines, the hypotenuses of these triangles, uniformly, in order to obtain an area-true representation in a quincunx grid. When $x = 0$ of the beginning meridian $\lambda = 0$, the equations which apply for the eighth of the globe in the region $-\frac{\pi}{4} < \lambda < \frac{\pi}{4}$ and $\delta < \frac{\pi}{2}$ are:

$$y = \sqrt{\pi}\,\sin\frac{\delta}{2}\,;\quad x = \left(4\lambda\,\sin\frac{\delta}{2}\right) : \sqrt{\pi}$$

The grids in each two adjacent eighths of the globe are symmetrical to their common side.

The rectangular breaks of the N in a grid such as the projection No. 234 are especially irritating. One may avoid them, if one rounds off the corners, e.g., through quadrants. If N in Illustration 22 (plate I) represents the pole and each of the square sides AB and BC represent a quarter of the equator, then for area-equality of the globe octant NABC, the quadrant side must be $\sqrt{\frac{\pi}{2}}$. Now one could draw for C and a each a quadrant through B and N and let the quarter of the N-line δ consist of the two straight line segments DE and GH and the quadrant EFG about the center J. If one then takes the coordinates of the point E, ND = y; DE = x, that is, the circle radius JE = y - x, then the stipulation for true-area between the area NDEFGH and its original on the globe yields the equation

$$\pi \sin^2 \frac{\delta}{2} = 2xy - x^2 + (y - x)^2 \frac{\pi}{4}$$

and, between x and y, the equation for the circle NEB:

$$y^2 + \left(\sqrt{\frac{\pi}{2}} - x\right)^2 = \frac{\pi}{2}$$

x and y are given in their dependence on through both equations, so that the N can be drawn. The following little table gives correlated values for the case of a uniform globe of 40 mm radius.

[See table on page 167a following.]

It is quite complicated to space the thus area-equal zones exactly area-true through the H. Illustration 23 (plate VIII) gives the grid of such a star map of 15° to 15° in length and width. The arcs are drawn dotted on an octant of the globe; the straight and arcing segments of the N converge on them. The H are simply drawn as pairs of straight lines which divide the equator uniformly. The deviations from complete true-area are thus only slight, and the area-equality of the zones is

δ 0°	15°	30°	45°	60°	75°	90°
y (mm) 0	10,08	20,00	29,80	36,72	43,72	50,13
x (mm) 0	1,02	4,10	9,10	16,05	24,30	50,13

exactly preserved in this projection SNo. 235. If x_o and y_o are connected in this projection according to the equations

$$\pi \sin^2\frac{\delta}{2} = 2x_0 y_0 - x_0^2 + (y_0 - x_0)^2\frac{\pi}{4};$$

$$y_0^2 + \left(\sqrt{\frac{\pi}{2}} - x_0\right)^2, = \frac{\pi}{2}$$

in the first quarter of the northern hemisphere, we find the equation for N: $y = y_o$ so long as x `x_o and

$$(x - x_0)^2 + (y - y_0)^2 = (y_0 - x_0)^2$$

for $x > x_o$, the equation for the H-straight lines:

$$\mathrm{tg}\ \alpha = 4\lambda : \pi$$

For the last order of our system, the *many-surface- projections (Polyhedral projections),* one imagines the uniform globe as surrounded by a polyhedron, with a piece of the globe represented on each of its surfaces. Then the polyhedron is to be cut up, adjacent surfaces laid next to each other and so spread out for a map surface. Insofar as one expects to produce an uninterrupted, coherent world map, the reproduction rule must make sure that that same line from the globe is reproduced exactly the same where an edge is formed by two adjacent polyhedron surfaces. This is not the case with the so-called

"Prussian Polyhedral Projection" [104] nor with the *"modified polyconical projection"* proposed by Lallemand[105] for the international world map with scale of 1 : 1,000,000. We shall leave such manners of reproduction our of our system, since they actually give no uniform reproduction of the whole globe on a plane surface. They are really reproduction of small pieces of the globe on *very many single leaves* only a few of which can be considered as adjacent to each other at given times, as long as the resulting interruptions or overlappings are slight enough not to disturb.

In contrast, uniform reproductions on a polyhedron of a low number of surfaces n are also possible; the uninterrupted coherence of the whole world map can be exactly determined mathematically by means of the reproduction rule. We have included two examples of this sort. SNo. 236 is the *perspective of the globe from its center point onto a hexadedron which is contiguous with it at six points.* The points are

($\phi = 0°$, $\lambda = 10°W$ and $170°E$)

($\lambda = 80°E$, $\phi = 60°N$ and $30°S$)

($\lambda = 100°W$, $\phi = 30°N$ and $60°S$)

The Coast and Geodetic Survey in Washington[106] has used this possibility in such a manner as to reproduce North America undistorted on the map, which, however, is done at the cost of other parts of the earth. Africa is divided into three, Antarctica into two pieces widely divided from each other. The southern parts of both Indian peninsulae are far separated from Asia, as are the Pyrenean peninsula and half of Franch from Europe; and South America is rectangularly slit open down to its center.

The world map SNo. 237 (Illustration 24, plate VII) has been projected according to almost the same idea of reproduction, but it avoids all distortion of the lands of the earth. It is the *perspective of the globe from its center point onto the walls of an eight-cornered rectangular box* with quadratic lid and floor, which is contiguous with the globe at the two poles of the earth, while the four side walls are parallel to the meridian planes $\lambda = 20^{\circ}$ E and $\lambda = 70^{\circ}$ W at a distance from them equal to one and a half times the global radius.

The four meridian slits take their courses along the meridians 65° and 155° E, 25° and 115° W from the corners of the square surrounding the north pole from $25^{\circ}15'$ toward the south. They are to be drawn in two squares between 65° and 155°E and between 155°E and 115°W to $25^{\circ}15'$S. In the two other squares, one draws them somewhat further toward the south, in order to attach the southern ends of Africa and South America uninterrupted in an unaltered projection. Every great circle follows a straight line course on each of the six leaves of the two projections No. 236 and 237, since the reproduction rule for the center-perspective projection (SNo. 1) applies for each of the leaves. The circles of latitude in the single leaves of projection No. 237 are circles on the lid and floor of the box, hyperbolae on the side walls; in projection No. 236 they are arcs of ellipses, parabolae and hyperbolae. The equations for lid and floor in No. 237 are as for the center-perspective earth-axical projection, $r = \mathrm{tg}\ \delta$; $\alpha = \lambda$, and on the side walls (λ calculated from the central meridian of the region), $y = 1.5\ \mathrm{tg}\ \lambda$; $x = 1.5 \sec\lambda\ \mathrm{tg}\ \phi$.

SECTION III: A DIAGRAM OF PROPERTIES FOR MAP PROJECTION

In our system table, each projection proves to be a certain combination of characteristic properties. The table makes it possible to judge just which properties may be combined. On the other hand, it does not discern which combinations of properties are impossible. This diagram gives information about both questions, about possibilities of combinations, as well as about the mutual exclusion of properties. Family V is not included in this examination, since the grids of that family exhibit various combinations of properties in particular parts of the map in a manner which is not uniform.

We examine the combination possibilities of the following fourteen properties:

Fourteen Characterizing Properties of Maps

Abbreviation and Designation	Definition
1. M = centrally-circular	The N are same-centered circles. To this group belong the whole of families I and II, nothing from family III, the cylinder-circular projections of family IV.
2. D = doubly-symmetrical	The grid is symmetrical to the basic line and to the central H. To this group belong the whole of families II and III, nothing from families I and IV.

Abbreviation and Designation	Definition
3. N = secondary circle-spaced	All N are equal-spaced. To these belong: Family I, No. 1 - 73; Family II, No. 83 - 152; and single projections from family III and IV.
4. Z = zenithal (cellular)	Z combines with M and N the property that all H are congruent with their divisions. To these belong: Family I, No. 1 - 58; Family II, No. 83 - 97; the cylindrical projections of family IV; nothing from family III. Z is a subdivision of M and N.
5. K = generally conical	Subdivision of Z with straight H. To these belong the projections named under Z, except No. 53 - 58.
6. R = radial (azimuthal)	Subdivision of K. All N are full circles. To these belong family I, No. 1 - 30, and none from other families.

Abbreviation and Designation	Definition
7. H = prime-point map	Subdivision of M. A prime-point is represented as center of the N. To these belong numerous projections from Family I, No. 92 - 99; No. 157 from Family II; No. 197, 198 from Family IV; Nothing from Family III.
8. A = interval-true	The N are either equal-intersecting circles or congruent, parallel displaced curves. Their intervals are equal to the arc intervals of their originals on the uniform globe. A is found in all the families except for Family III.
9. F = area-true	Each piece of surface of the map is area-equal to its original on the uniform-globe. F is found in all families.
10. W = shape-true	All curves of the map intersect at the same angles as on their originals on the uniform globe. W is found in all families.

Abbreviation and Designation	Definition
11. O = orthodromic (straight-directional)	All straight lines of the map are pictures of great circles of the globe. O is not possible in Family II.
12. L = loxodromic (course-linear)	All straight lines of the map are pictures of course-equals (loxodromes) of the globe. L is not possible in Families I and III.
13. G = counter-azimuthal	The straight lines through a shape-true map-point are pictures of the azimuth-equals which have the original of this point as direction-point. G is possible in all families.
14. P = perspective	The map is the perspective picture of the uniform globe on a cone which has the same axis and which can also deteriorate into a plane or cylinder, from a sighting-point on the common axis. P is not possible in Family III, but is possible in all other Families.

These fourteen properties can yet be ordered in two basically different groups. The first eight, M, D, N, Z, K, R, H, A, apply for a particular grid of H and N on the map; they are *grid properties*. On the other hand, the properties F, W, O, L, G and P apply for the map no matter which grid of H and N we want to consider for its construction; for, the areas, angles, and straight lines of the map, as well as its perspective orientation, do not change, even if we change over to another grid of H and N. These properties, 9 - 14, are *independent of the grid*. Thus, when one wants to combine several stipulations of the 1 - 8 group in a projection, one must do so with respect to the same grid. On the other hand, when fulfilling stipulations of the group 9 - 14 and at the same time from group 1 - 8, the choice of grid remains completely free. One must only state which grid is supposed to apply for the map as a means of designating such combination of properties. If, for example, we think of Hammer's center-interval-true counter-azimuthal projection (SNo. 78), then we naturally think of the polar grid with respect to its counter-azimuthality (G) since the concept of the counter-azimuth is connected in one's imagination with the meridians. But the polar grid in this map is not centrally circular (M) or interval-true (A); it is, rather, the grid of the shape-true direction-point of azimuth-equals reproduced as straight lines. Or, if the grid of a perspective (P) projection is supposed to be neither centrally-circular (M) nor doubly-symmetrical (D), then one is obviously not talking about its prime-point, which is, of course, always supposed to be centrally-circular; but, no doubt, the lack of center-circularity and double-symmetry applies for the polar grid of an oblique-axical perspective projection.

The Property Table

Everything else will become evident in the following property table, which states all 84 really possible combinations from among the 14 properties and enumerates for each such combination which projections from our system table are thusly characterized. Each possible combination is shown in this table by crosses (+) underneath that property fulfilled in that projection. The lexicographic arrangement according to *field numbers* (FNo.) 1 - 84 is carried out in such a manner that that combination comes first which has crosses in the column more to the left.

Table to Diagram of Properties of Map Projections

[See pages 182a and 182b following.]

Parts of the System Table	Field No. in Diagram	Properties of Grids	Properties Independent of Grids

[Last column] Examples from the System

	Examples:
Family II	No. 94,95 Etzlaub-Mercator
Branch A, Subbranch A	No. 94 Prime-point grid of transverse-axical Mercator projection
	No. 96 Mercator map, projected perpendicular to map plane on plane through equator
	No. 93 Wetch, cylinder-perspective projection
	No. 97 Special cylindrical projection, No. 92
	No. 90, 91 flat maps
	No. 87-89 Lambert and Behrman
	No. 83-86 Circle-perspective cylindrical projections
Family II	No. 98 special cylinder-circular projection
Branch A, Subbranch B	No. 99 Special cylinder-circular projection
	No. 100 Mercator (Sauson)
	No. 101,105,108,111,114,117,120,123,125, 128,130 after Apianus II, Winkel, Eckert, Donis, Wagner
	No. 102, 131-149 after Eckert, Collignon, Mollweide, Nell, Hammer, Wagner, Craster, Prepetit-Foulaut, Bourdin

	No. 103-4, 106-7, 109-10, 115-16, 118-19, 121,123-24,126-27,129,150-152
Family II, Branch B, Subbranch IV, No. 215,216	No. 157 special cylinder-cirular projection
	No. 215 in the case $\phi_0 = 0$, polar grid of special counter-azimuthal projection
	No. 153, 154 Apianus I
	No. 216 in the case $\phi_0 = 0$, polar grid of special counter-azimuthal projection
	No. 3 Polar grid of the transverse-axical distant-perspective projection
	No. 155, 156 Glareanus, van der Grinten, No. III
Family I, Branch A Subbranch A Order a, Class I	No. 24 Mercator (Postel)
	No. 23 Lambert
	No. 2 All-circular, Prime-point grid (Gnomonic)
	No. 1 Center-perspective, prime-point grid (gnomonic)
	No. 3-21 Outer-perspective, prime-point grid (external)
	No. 22, 25-27 Radial introening after Lidman, Breusing, Airy, James and Clark
	No. 30 radial ring map Maurer
	No. 28,29 Radial ring maps Hammer, Maurer
	No. 31 Special ring map
Family I, Branch A Subbranch A, Order a, Class II	No. 41-43 after Schjerning
	No. 44 Lambert
	No. 39,40 Lambert-Gaub
	No. 32-27, 197, 198 Murdoch, Braun
	No. 38 Special case of zone-perspective projections

No. 47-49 Ptolemaus, De L'Isle, Murdoch I

No. 51-52 Albers

No. 45-46 Zone-perspective, Murdoch III

Family I, Branch A
Subbranch A, Order b

No. 54, Special Projection, All H congruent arcs

No. 53, special projection, all H congruent arcs, Wiechel

No. 224, Projected on plane perpendicular to map plane, not parallel to equator or to central-meridian

No. 55 special projection all H congruent arcs

No. 56 special projection all H congruent arcs

No. 57 special projection all H congruent arcs

No. 58 special projection all H congruent arcs

Family I, Branch A,
Subbranch B, Family IV,
No. 207-214

No. 63 after Stab

No. 61, 67-69 Schjerning No. V, III, II

No. 65,72,73 Nell, Maurer

No. 66, special case of Schjerning No. V

No. 59 Mercator (Bonne)

No. 60 and 208, 210, 211

No. 61, 70, 71, 212, 213 Nell, Eckert, Collignon

No. 267 Mercator Maps Projected onto a plane through the equator from a sighting point in the finite

No. 62, 209, 214

Family I, Branch B
Family IV, No. 215,216

No. 78 Hammer, Grid of a special point

No. 74, 77 special radial-circular, non-zenithal grids

	No. 79, 80 grid of special point
	No. 75 special radial-circular, non-zenithal grid
	No. 215, when $\phi_0 \gtrless 0$
	No. 81, 82 simplified conical projection
	No. 216, when $\phi_0 \gtrless 0$
	No. 76 special radial-circular, non-zenithal grid
Family III	No. 176, 178 ordinary poly-conic and secondary-circular all-globular projection
	No. 219, when $\phi_0 = 0$
	No. 179, 183, 184, 186 area-true poly-conic and affined or Aitoff-ized area-true projection
	No. 189 Littrow-Maurer
	No. 170 polar or transverse-axical all-circular projection
	No. 169, 171-173, 181, 190-194 shape-true circle-grids and projections by Fiorini, August, Eisenlohr, Guyou
	No. 1 Polar grid of a transverse-axical center-perspective projection, affined
	No. 196 Doubly-counter-azimuthal; No. 200, when $\phi_0 = 0$, Craig; No. 78-80 polar grids, when $\phi_0 = 0$
	No. 3-21 polar grid of transverse axical outer-perspective projections
	No. 158-168, 174, 175, 177, 180, 185, 187, 188, 195 Circle-grids, rectangular poly-conic projection, Aitoff, prime-circular all-globular, Schmidt, No. 199 for $\phi_0 = 0$

Family IV

No. 205 special projection, N parallel arcs

No. 206 special projection, N parallel arcs

No. 201, 202 special projection, N parallel arcs

No. 219 when $\phi_0 \gtrless 0$

No. 203,204 special projections, N parallel arcs

No. 2 polar grid of oblique-axical all-circular projections

No. 217, 218, 220-223 shape-true circle-grids, Peiree, Mercator, reproduced by reciprocal rays

No. 1 polar-grid of oblique-axical, center-perspective projections

- same as 80, affined

No. 225 projected from sighting-point within finite onto plane which is neither parallel to equator not to center-meridian

No. 78-80, 200, when $\phi_0 \gtrless 0$

No. 3-21 polar grid of transverse-axical projections

No. 199 Schoys radial-directional grid ($\phi_0 \gtrless 0$)

Tabelle zum Begriffsschaubild der Weltkartenentwürfe

Teile der Systemtabelle	Feld-Nr. im Schaubild	Eigenschaften der Netze: M	D	n	z	K	r	R	a	Netzfreie Eigenschaften: f	w	o	l	g	p	Beispiele aus dem System
Stamm II, Ast 9, Zweig A	1	+	+	+	+	+		+			+		+			Nr. 94, 95 Einlaub-Merkator
	2	+	+	+	+	+		+			+					Nr. 91 Hauptpunktnetz querachsigen Merkatorentwurfs
	3	+	+	+	+	+		+					+			Nr. 96 Merkatorkarte, senkrecht zur Kartenebene projiziert auf Ebene durch den Äquator
	4	+	+	+	+	+		+							+	Nr. 93 Wetch, säulensichtiger Entwurf
	5	+	+	+	+	+		+								Nr. 97 Besonderer säuliger Entwurf, Nr. 92
	6	+	+	+	+	+			+							Nr. 90, 91 Plattkarten
	7	+	+	+	+	+				+						Nr. 87--89 Lambert und Behrmann
	8	+	+	+	+	+										Nr. 83--86 Kreissichtige säulige Entwürfe
Stamm II, Ast 9, Zweig B	9	+	+	+				+		+						Nr. 98 Besonderer säulenkreisiger Entwurf
	10	+	+	+				+								Nr. 99 " " "
	11	+	+	+					+	+						Nr. 100 Merkator (Sanson)
	12	+	+	+					+							Nr. 101, 105, 108, 111, 114, 117, 120, 123, 125, 128, 130 nach Apianus II, Winkel, Eckert, Donis, Wagner
	13	+	+	+						+						Nr. 102, 131--149 nach Eckert, Collignon, Mollweide, Nell, Hammer, Wagner, Craster, Prépetit-Foucaut, Boudin
	14	+	+	+												Nr. 103/04, 106/07, 109/10, 112/13, 115/16, 118/19, 121, 123/24, 126/27, 129, 150--152
Stamm II, Ast 9 Stamm IV, Nr. 215, 216	15	+	+					+								Nr. 157 Besonderer säulenkreisiger Entwurf
	16	+	+						+					+		Nr. 215 im Fall $\varphi_0 = 0$. Polnetz besonderen gegenachsigen Entwurfs
	17	+	+						+							Nr. 153, 154 Apianus I
	18	+	+											+		Nr. 216 im Fall $\varphi_0 = 0$. Polnetz besonderen gegenachsigen Entwurfs
	19	+	+												+	Nr. 3 Polnetz des querachsigen fernsichtigen Entwurfs
	20	+	+													Nr. 155, 156 Clareanus, van der Grinten Nr. III
Stamm I, Ast 9, Zweig A Ordnung a, Klasse I	21	+		+	+	+	+	+	+							Nr. 21 Merkator (Postel)
	22	+		+	+	+	+	+		+						Nr. 23 Lambert
	23	+		+	+	+	+	+			+				+	Nr. 2 Allkreisig, Hauptpunktnetz (Stereographisch)
	24	+		+	+	+	+	+				+			+	Nr. 1 Mittensichtig, Hauptpunktnetz (Gnomonisch)
	25	+		+	+	+	+	+							+	Nr. 3--21 Außensichtig, Hauptpunktnetz (Extern)
	26	+		+	+	+	+	+								Nr. 22, 25--27 radlich vermittelnd nach Littrow, Breusing, Airy, James und Clarke
	27	+		+	+	+	+		+							Nr. 30 radliche Ringkarte Maurer
	28	+		+	+	+	+			+						Nr. 28, 29 radliche Ringkarten Hammer, Maurer
	29	+		+	+	+	+									Nr. 31 Besondere Ringkarte
Stamm I, Ast 9, Zweig A Ordnung a, Klasse II	30	+		+	+	+		+	+							Nr. 41--43 nach Schjerning
	31	+		+	+	+		+		+						Nr. 44 Lambert
	32	+		+	+	+		+			+					Nr. 39, 40 Lambert-Gauß
	33	+		+	+	+		+								Nr. 32--37, 197, 199 Murdoch, Braun
	34	+		+	+	+		+					+			Nr. 38 Sonderfall zonensichtiger Entwürfe
	35	+		+	+	+			+							Nr. 47--49 Ptolemäus, De l'Isle, Murdoch I
	36	+		+	+	+				+						Nr. 51--53 Albers
	37	+		+	+	+										Nr. 45/46 zonensichtig, 50, Murdoch III
Stamm I, Ast 9, Zweig A Ordnung b	38	+		+	+			+	+							Nr. 54 Besonderer Entwurf. Alle K kongruente Kreisbogen
	39	+		+	+			+		+						Nr. 53 Wiechel " " " "
	40	+		+	+			+					+			Nr. 234 Merkatorkarte, senkrecht zur Kartenebene projiziert auf Ebene, die weder zum Äquator noch zum Mittelmeridian parallel ist
	41	+		+	+			+								Nr. 55 Besonderer Entwurf. Alle K kongruente Kreisbogen
	42	+		+	+				+							Nr. 56 " " " " " "
	43	+		+	+					+						Nr. 57 " " " " " "
	44	+		+	+											Nr. 58 " " " " " "

Tabelle zum Begriffsschaubild der Kartenentwürfe (Fortsetzung)

Teile der Systemtabelle	Feld-Nr. im Schaubild	Eigenschaften der Netze								Netzfreie Eigenschaften						Beispiele aus dem System
		M	D	n	z	K	r	H	s	f	w	o	l	g	p	
Stamm I, Ast ?, Zweig B; Stamm IV, Nr. 207—214	45	+		+				+	+	+						Nr. 63 nach Stab
	46	+		+				+	+							Nr. 64, 67—69 Schjerning Nr. V, III, II
	47	+		+				+		+						Nr. 65, 72, 73 Nell, Maurer
	48	+		+				+								Nr. 66 Sonderfall von Schjerning Nr. V
	49	+		+					+	+						Nr. 59 Merkator (Bonne)
	50	+		+					+							Nr. 60 u. 208, 210, 211
	51	+		+						+						Nr. 61, 70, 71, 212, 213 Nell, Eckert, Collignon
	52	+		+									+			Nr. 207 Merkatorkarten, projiziert auf Ebene durch den Äquator aus Augpunkt im Endlichen
	53	+		+												Nr. 62, 209, 214
Stamm I, Ast ?, Stamm IV, Nr. 215, 216	54	+						+	+					+		Nr. 78 Hammer, Netz des Sonderpunkts
	55	+						+	+							Nr. 74, 77 Besondere radkreisige, nicht zenitale Netze
	56	+						+						+		Nr. 79, 80 Netz des Sonderpunkts
	57	+						+								Nr. 75 Besonderes radkreisiges, nicht zenitales Netz
	58	+							+					+		Nr. 215, wenn $\varphi_2 \gtrless 0$
	59	+							+							Nr. 81, 82 Vereinfachter Kegelentwurf
	60	+												+		Nr. 216, wenn $\varphi_2 \gtrless 0$
	61	+														Nr. 76 Besonderes radkreisiges, nicht zenitales Netz
Stamm III	62		+	+												Nr. 176, 178 Gewöhnlicher polykonischer und nebenkreisig allglobularer Entwurf
	63		+							+				+		Nr. 219, wenn $\varphi_2 = 0$
	64		+							+						Nr. 179, 183, 184, 186 Flächentreuer polykonischer und affinierter oder altollierter flächentreuer Entwurf
	65		+								+			+		Nr. 180 Littrow-Maurer
	66		+								+				+	Nr. 170 Polnetz querachsigen allkreisigen Entwurfs
	67		+								+					Nr. 169, 171—173, 181, 190—194 Winkeltreue Kreisnetze und Entwürfe von Fiorini, August, Eisenlohr, Guyou
	68		+									+			+	Nr. 1 Polnetz querachsigen, mittensichtigen Entwurfs
	69		+									+				Nr. 152 Polnetz querachsigen, mittelsichtigen Entwurfs, affiniert
	70		+											+		Nr. 195 Doppelt-gegenazimutal; Nr. 200, wenn $\varphi_2 = 0$, Craig; Nr. 78—80 Polnetze, wenn $\varphi_0 = 0$
	71		+												+	Nr. 3—21 Polnetz querachsiger außensichtiger Entwürfe
	72		+													Nr. 158—168, 174, 175, 177, 180, 185, 187, 193, 195 Kreisnetze, rechtschnittiger polykonischer Entwurf, Aitoff, hauptkreisig-allglobular, Schmidt, Nr. 190 für $\varphi_2 = 0$
Stamm IV	73			+					+							Nr. 205 Besonderer Entwurf. Die N parallele Kreisbogen.
	74			+												Nr. 206 „ „ „ „ „ „
	75								+							Nr. 201, 202 „ „ „ „ „ „
	76									+				+		Nr. 219, wenn $\varphi_2 \gtrless 0$
	77									+						Nr. 203, 204 Besondere Entwürfe; die N parallele Kreisbogen
	78										+				+	Nr. 2 Polnetz schiefachsiger allkreisiger Entwürfe
	79										+					Nr. 217, 218, 220—223 winkeltreue Kreisnetze; Peirce; Merkator, abgebildet durch Kehrwertstrahlen
	80											+			+	Nr. 1 Polnetz schiefachsiger mittensichtiger Entwürfe
	81											+				— „ „ „ „ , affiniert
	82												+			Nr. 225 Merkator aus Augpunkt im Endlichen projiziert auf Ebene, die weder zum Äquator noch zum Mittelmeridian parallel ist
	83													+		Nr. 78—80, 200, wenn $\varphi_0 \gtrless 0$
	84														+	Nr. 3—21 Polnetz schiefachsiger außensichtiger Entwürfe
	85															Nr. 199 Schoys radstrahliges Netz (\varphi_2 \gtrless 0)

Clarification of the Diagram of Properties

A graphic presentation of the contents of this table is offered in the Diagram of Properties, Illustration 25 (plate X). It shows 15 areas, designated by means of the varying markings of its border line and the identifying letters M,D,N,Z,K,R, H,A,F,W,O,L,G, and P. (There are two areas for O.) The areas for a top have a tube-shape and fine border lines; the others, M to H, have curved, often indented, border lines with various kinds of signature. The 15 border lines enclose or exclude each other or touch or intersect each other in many various ways, so that 84 fields with identifying *field numbers* arise, each of which is enclosed by at least one of the 15 properties. Each field represents the combination of those properties inside of whose identifying lines it occurs; thus the field corresponds to that line in the property table on which those properties are marked with a cross.

For example, field 39 lies inside the border lines M,N,Z, H, and F, but outside all others. Thus it corresponds to the zenithal area-true prime-point-maps, which, however, are not generally-conical. As an example for this field under FNo. 39, the table gives the Wiechel Projection, SNo. 53. Field 49 lies inside the border lines M,N,A,F but outside all others. The corresponding projections are, then, secondary-circle-spaced, centrally-circular, area-true and interval-true, but they are neither prime-point-maps nor zenithal. For this field, the property table gives under FNo. 49 the example of the Mercator-Bonne Projection, SNo. 59. Field 72 refers to projections which are doubly-symmetrical (D) but otherwise exhibits none of the other 13 properties, as do, for example, the globular projection, SNo. 158, and many others.

The diagram allows one to get a comprehensive view of a great number of facts about cartography with one glance. One sees immediately that true-shape cannot be combined with either true-area or equidistance, although with every other of our 14 map properties; that loxodromic maps can be neither radial nor perspective, neither straight-directional nor counter-azimuthal, but can exhibit all 9 other properties. Counter-azimuthality can be combined with equidistance, true-area and true-shape; counter azimuthal maps can also be centrally-circular, double-symmetrical and prime-point-maps, but neither secondary-circle-spaced nor zenithal.

That many special combination possibilities of the last four properties, O, L, G, P, with others are based on the following: In perspective projections (P), the *polar grid* can exhibit, according to whether its axical inclination is earth, transverse or oblique, very different forms of sharply changing regularity (FNo. 4, 19, 23, 24, 25), even to the point of lacking any other property (FNo. 84); the doubly-symmetrical, cylinder-circular ring map polar grid of the transverse-axical, distant-perspective projection (FNo. 19) is especially noteworthy. In three instances (all-circular projections), the perspective polar grids are shape-true (FNo. 23, 66, 78), and in three others they are (central-perspective projection) straight-directional (FNo. 24, 69, 81). One does not obtain perspective projections, but rather straight-directional projections (FNo. 84) by means of colinear transformation of the polar grid of a transverse-axical, center-perspective projection. The variety of loxodromic projections results from colinear reproductions of Mercator projections; and it has already been pointed out on p. that very different sorts of properties can be affixed to counter azimuthality, which is actually only an equation for the coordinates of a polar grid.

Making excursions from field to field in the diagram is very stimulating and educational! Neighboring fields, which are divided by only one simple border line, are different by only one property, so that one can begin at one sort of projection and wander through gradually changing levels, arriving, then, at projections of completely properties. If, for example, one goes from field 1 (e.g., the Etzlaub-Mercator-Map), in which 8 stipulations (M,D,N,Z,K,H,W,L) are fulfilled, and moves always toward the left, then he finds that in field 2 (e.g., Lambert Gauss = transverse-axical Mercator projection) only the course-linearity (l) is lost, since the loxodromes in that field are not straight lines. Then the shape-true conical projection follow in field 32 with the loss of double-symmetry (D), whereupon in field 23 the now all-circular projection becomes both radial (r) and perspective (p) and still maintains true-shape (w). This true-shape is then lost in the neighboring field 25 (an outer-perspective projection); but in field 24 (a centrally-perspective projection) one can trade in straight-directionality (O) for true-shape. Now follows field 26, still radial (r), but no longer perspective (P). Equidistance (A) joins up in field 21 (Mercator-Postel Projection), which in field 22 and 23 is substituted for by true-area. Field 22 yields, as all fields up to now, prime-point-maps H (Lambert's projection SNo. 23), but field 28 is for ring-maps. Radial ring maps without special properties follow in field 29; in field 37 are yet conical, and in field 44 still more conical, although still zenithal, ring maps, whereas there is only center-circularity (M) in fields 53 and 61, combined in field 53 with being secondary-circle-spaced (N).

In a similar manner, a hike through the tube-shaped areas of the interval-true (a) or area-true (f) projections is interesting.

The field numbers are also included in the large system table in the second column. The relationship of the system with the diagram becomes especially clear in the following manner: The field area of the *Family* I (SNo. 1 - 82, FNo. 21 - 61) is marked with brown in the diagram, and the area of its branch A (subbranch A = SNo. 1 - 58, FNo. 21 - 44; subbranch B = SNo. 59 - 73, FNo. 45 - 53) with two kinds of brown oblique striations, the area of its branch B (SNo. 74 - 82, FNo. 54 - 61) with brown borders in two parts of the surface. The field area of *Family* II is designated by red color, in the area of its branch A (subbranch A = SNo. 83 - 97, FNo. 1 - 8; subbranch B = SNo. 98 - 152, FNo. 9 - 14) with two kinds of red oblique striation, and in the area of its branch B (SNo. 153 - 157, FNo. 15 - 20) with red border. *Family III* (SNo. 158 - 196, FNo. 62 - 72) is designated by means of the red-brown border of two parts of the surface of the diagram. Its branches are not singled out, since row-circularity has not been found useful among the 14 properties of our diagram. The outer-most border fields (FNo. 73 - 84) belonging to *Family IV*, and the FNo. 85 lying outside all border lines and not fulfilling any of our 14 properties, are not designated by any color.

The last column of the table to the diagram contains all 225 SNo.'s of families I - IV of our system table. Each fulfills at least one of the 14 characterizing properties of the diagram with the exception of the only radial-directional grid of projection SNo. 199 by Schoy, which has been assigned to the marginal space FNo. 85 in the general case $\phi_0 \gtrless 0$

Index of Germanizations

(On the left are the German germanizations, from the Latin or Greek derived term to the "German" word. On the right is the English literal equivalent to this process. The translator has reordered Maurer's list so that the English is in alphabetical order.)

General

Zentrum = Mitte	center = center, middle
koaxial = gleichachsig	coaxial = equal-axical, of the same axis
konzentrisch = gleichmittig	concentric = equal-centered, same-centered
Konus = Kegel	cone = cone
Zylinder = Saule	cylinder = cylinder (or pillar)

aquidistant = gleichabstandig	equidistant = equal-interval, equidistant
homogen = gleichteilig	homogeneous = of equal parts, equal-spaced
orthogonal = rechtschnittig	orthogonal = rectangular
parallel = gleichlaufend	parallel = parallel
Projektion = Entwurf	projection = projection, reproduction, picture
Quincunx = Funfform	quincunx = quincunx, five-form
Radius = Halbmesser	radius = radius
Sektor = Kreisausschnitt	sector = sector, sector of a circle
Segment = Kreisabschnitt	segment = segment, segment of a circle
Symmetrie = Spiegelgleichheit	symmetry = symmetry
Tangentialzylinder = Beruhrsaule	tangential cylinder = tangential cylinder, contiguous cylinder
Transformation durch reziproke Radien = Umwandlung durch Kehrwertstrahlen	transformation by means of reciprocal radials = transformation by means of reciprocal rays

Designation of Lines

Aquideformaten = Verzerrungsgleichen	equiform = distortion-equal, of same distortion
Horizontalkreise = Nebenkreise	horizontal circles (parallels) = secondary-circles (abbr.: N)

Loxodromen = Kursgleichen	loxodrome = course-equal
Orthodromen = Gro kreise	orthodrome = great circle
Orthogonale Trajektorien = Rechtschnittgleichen	orthogonal trajectory = rectangular-intersection-equals
Parallelen = Breitenkreise	parallels = circles of latitude

Inclination of Projection

horizontal (zenithal) = schiefachsig	horizontal (zenithal) = oblique-axical
normal (Polarprojektion, Aquatorprojektion) = erdachsig	normal (polar projection, equator projection) = earth-axical
transversal (Aquatorialprojek-tion, Meridianprojektion = querachsig	transversal (equatorial projections, meridian-projections) = transverse-axical

Designation of Projections

achsial = mittkreisig	axial = axial, centrally-circular, center-circular
azimuthal = radlich (radkrei-sig und radstrahlig)	azimuthal = radial (radial-circular and radial-directional)
konform (autogonal, isogonal, orthomorph) = winkeltreu	conform (autogonal, isogonal, ortho-morphic) = shape-true
konisch = kegelig (kegelkreisig und kegelstrahlig)	conical = conical (conical-circular, conical-radial

zylindrisch = saulig	cylindrical = cylindrical
doppelsymmetrisch = doppelspiegelig	double-symmetrical = double-symmetrical, doubly-symmetrical
aquidistant = abstandstreu (punktabstandstreu und kreisabstandstreu)	equidistant = interval-true, distance-true (point-interval-true and circle-interval-true)
aquivalent (isomer, authalisch, homalographisch) = flachentreu	equivalent (isomeric, auto-, homalographic) = area-true
extern = au ensichtig	external = external-perspective, outer-perspective
geometrisch einfach definiert = allgemein kegelig	geometric, simply defined = generally-conical
gnomonisch (Zentralsperspektive) = mittensichtig	gnomonic (central-perspective) = center-perspective, central-perspective
isozylindrisch = flachentreu saulig	isocylindrical = area-true cylindrical
isopharischstenoter = flachentreu kegelig	isopheric stenoter = area-true conical
loxodromisch = kurslinig	loxodromic = loxodromic, course-linear
orthodromisch = geradwegig	orthodromic = orthodromic, straight-directional
orthographisch = fernsichtig	orthographic = orthographic, distant-perspective

perspektivisch = sichtig (ebensichtig, kegelsichtig, saulensichtig)	perspective = perspective (plane-perspective, cone-perspective, cylinder-perspective)
polykonisch = allkegelig	poly-conic = all-conical
quinkunktial = funfformig	quincuncail = quincuncial, five-form
unecht konisch (konventionelle Kegelprojektion) = kegelkreisig	untrue conical (conventional conical projection) = conical-circular
unecht zylindrisch (konventionelle Zylinderprojektion) = saulenkreisig	untrue-cylindrical (conventional cylindrical projection) = cylindrical-circular, cylinder-circular
zenital = facherig	zenithal = zenithal, cellular

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25	X	Diagram of properties of map projections

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(For translation of notes containing remarks,

see pages 197 and 198 following.)

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[3]) M. Groll, Kartenkunde I, Projektionen, hrg. von O. Graf. 2. Aufl.; Sammlung Göschen 1922, S. 261.

[4]) E. Hammer, Z. f. V. 1924, S. 363.

[5]) H. Maurer, Z. f. E. 1922, S. 115; Z. f. V. 1925, S. 142.

[6]) Maurer, Pet. M. 1914, II, S. 64.

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[8]) H., S. 188.

[9]) T.-H., Vorwort S. V.

[10]) Ha., Titel des Buches.

[11]) Schoy, A. d. H. 1913, S. 33.

[12]) Zöppritz: Leitfaden der Kartenentwurfslehre, Leipzig 1884, S. 27. „Projektionen, die zenitale genannt werden, weil an allen Punkten gleichen Zenitabstandes von der Mitte dieselben Veränderungen stattfinden."

[13]) Ha., S. 9, Anmerkung. „In echtzenitalen Projektionen (z. B. der von Wiechel) sind die Linien gleicher Verzerrung Kreise um den Kartenmittelpunkt (die Horizontalkreisbilder)."

[14]) Maurer, A. d. H. 1905, S. 355—67.

[15]) Z.-B., S. 59. „Aus der Eigenschaft der Zenitalität folgt, daß alle Punkte eines gleichen Zenitabstandes δ, die, wie auf der Kugel auch auf der Karte auf einem Kreise liegen, die gleichen Verzerrungen erleiden."

[16]) G., S. 234. „Eine Abbildung heißt zenital, wenn alle Punkte P, die auf der Erdkugel gleichweit von einem gewissen Zentralpunkte A abstehen, auch auf der Karte auf dem Umfang eines Kreises liegen, dessen Zentrum die Projektion A′ jenes Punktes A ist, und wenn alle durch A gehende größte Kreise durch Gerade dargestellt werden, die sich in A′

[17]) [18]) Ha., S. 9. „Zu den nicht azimutalen Zenitalprojektionen -- die Parallelkreise durch konzentrische Kreise, die Meridiane aber nicht durch Radien der letzteren, sondern durch Kurven dargestellt — gehört der herzförmige Entwurf von Stab-Werner" (Nr. 63 unseres Systems). „Dieser Spezialfall könnte Veranlassung geben zur Trennung dieser nicht azimutalen Zenitalprojektionen in *unechte* (Umfang eines beliebigen Parallelkreises [allgemein Horizontalkreises] nicht durch den *ganzen* Umfang eines Kreises dargestellt, aber immer noch der Pol [allgemein Kartenmittelpunkt] Anfangspunkt der Streckenmessung auf den Meridianen), wozu der Stabsche Entwurf gehören würde, und in *echtzenitale*, die z. B. durch die Wiechelsche Projektion (Nr. 53 unseres Systems) repräsentiert wären."

[19]) Hammer, Pet. M. 1915. „So wäre es wohl besser, statt des seitherigen (auch von mir beobachteten) Gebrauchs, den Namen zenital solchen Abbildungen vorzubehalten, die das unendlich kleine Gebiet des Kartenmittelpunktes kongruent abbilden wie in der Projektion von Peirce (Nr. 218 unseres Systems) und in jeder azimutalen Projektion." (Sonderbarerweise wird aber dann den querachsigen Zylinderprojektionen, von denen anerkannt wird, daß sie auch das Flächenelement um den Pol kongruent abbilden, die Zenitalität abgesprochen.)

[20]) Maurer, Pet. M. 1914, II, S. 62—66; Z. f. E. 1919, S. 163—165; Pet. M. 1920, S. 57; 1922, S. 188.

[21]) Hammer, Pet. M. 1915, I, S. 97.

[22]) z. B. W. Immler, A. d. H. 1919, S. 22.

[23]) Z.-B., S. 86.

[24]) T.-H., S. 90, Anm.

[25]) T.-H., S. 149.

[26]) Z.-B., S. 141.

[27]) T.-H., S. 135.

[28]) Maurer: Kann die Winkeltreue in Einzelpunkten winkeltreuer Karten fehlen? (A. d. H. 1919, S. 212 bis 223.)

[29]) Z.-B., S. 187.

[30]) T.-H., S. 141.

[31]) T.-H., S. 116.

[32]) Z.-B., S. 120.

[33]) [illegible]

[34]) G., S. 14[illegible]

TEXT NOT REPRODUCIBLE

[35]) Z.-B., 121–37.
[36]) Z.-B., S. 200.
[37]) Nell, Pet. M. 1890, S. 93.
[38]) Hammer, Pet. M. 1900, S. 41.
[39]) Maurer, Pet. M. 1914, S. 64, Nr. XI.
[40]) Maurer, A. d. H. 1919, S. 77.
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[49]) z. B. Z.-B., S. 165.
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[52]) O. Winkel, G. Z. 1922, S. 178.
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[54]) K. H. W., S. 17.
[55]) K. H. W., S. 17 u. 23.
[56]) Craster, D. u. A., S. 165.
[57]) K. W. H., S. 5, Anm. 4.
[58]) Z.-B., S. 193, Fig. 112.
[59]) T.-H., S. 156.
[60]) B.-F., S. 231.
[61]) G., S. 163.
[62]) H., § 30, S. 133–48.
[63]) H., S. 79.
[64]) H., S. 79, Anm., u. 108, Anm.
[65]) Maurer, Z. f. E. 1922, S. 24.
[66]) Z.-B., S. 182, Fig. 110.
[67]) T.-H., S. 188, Nr. 140.
[68]) Thorade, A. d. H. 1919, S. 38.
[69]) Maurer, A. d. H. 1919, S. 218.
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[78]) Maurer, A. d. H. 1919, S. 76; 1921, S. 116; Pet. M. 1920, S. 57. Z. f. V. 1922, S. 14.
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[84]) Maurer, A. d. H. 1905, S. 125 u. 323.
[85]) Wedemeyer, A. d. H. 1919, S. 183, und Maurer, A. d. H. 1919, S. 278.
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[93]) D. u. A., S. 160, Fig. 74.
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[101]) Maurer, A. d. H. 1919, S. 20.
[102]) Maurer, Pet. M. 1911, S. 91.
[103]) Pet. M., Erg.-Heft 16, S. 67.
[104]) Z.-B., S. 149.
[105]) D. u. A., S. 62.
[106]) D. u. A., S. 52, Fig. 46.

Translation of notes containing remarks.

12 Zoppritz: Leitfaden der Kartenentwurfslehre, Leipzig 1884, S. 27. "...projections, which are called zenithal because changes take place at all points of equal zenith-distance from the center..."

13 Ha., S. 9, note. "In true-zenithal projections (e.g., the one by Wiechel) the lines of equal distortion are circles about the center-point of the map (pictures of horizontal circles).

15 Z.-B., S. 59. "It follows from the property of zenithality that all points of an equal zenithal-interval , which also lie on a circle of the map as on the globe, suffer the same distortion."

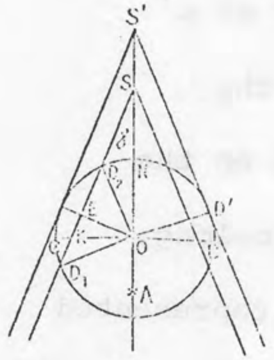
16 G., S. 234. "A projection is called zenithal if all points P, which are at an equal distance from a particular central point A on the globe, lie also on the circumference of a circle on the map, the center of which circle is a projection A' of that point A--even when all greatest circles passing through A are represented by straight lines which intersect at the same angle at A' as do the greatest circles at A."

17 18 Ha., S. 9. "The heart-shaped projection by Stab-Werner belongs to the non-azimuthal zenithal projections--the parallel circles represented by concentric circles, the meridians, however, not by radials of the latter, by by curves" (No. 63 of our system). "This special case could give rise to the dividing of these non-

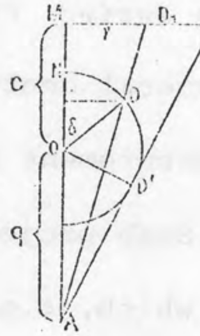
azimuthal zenithal projections into *untrue-zenithal* (circumference of an arbitrary parallel circle [a general horizontal circle] not represented by the *entire* circumference of a circle, but still the pole [general center point of the map] beginning point of the measurement of intervals on the meridians), to which group the Stab projection would belong, and *true-zenithal* projections, which, e.g., would be represented by the Wiechel projection" (No. 53 in our system).

[19] Hammer, Pet. M. 1915. "Thus, it would most likely be better, in place of the custom until now (I have also noted this custom), to reserve the name zenithal for such projections as reproduce the infinitely small region of the map center point congruently, as in Peirce's projection (No. 218 in our system) and in every azimuthal projection." (Strangely enough, this author then proceeds *not* to grant zenithality to the transverse-axical cylindrical projection which has just been recognized as reproducing the area around the pole congruently.)

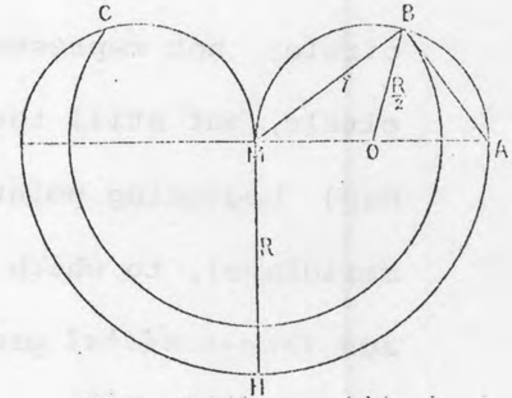
Plate I



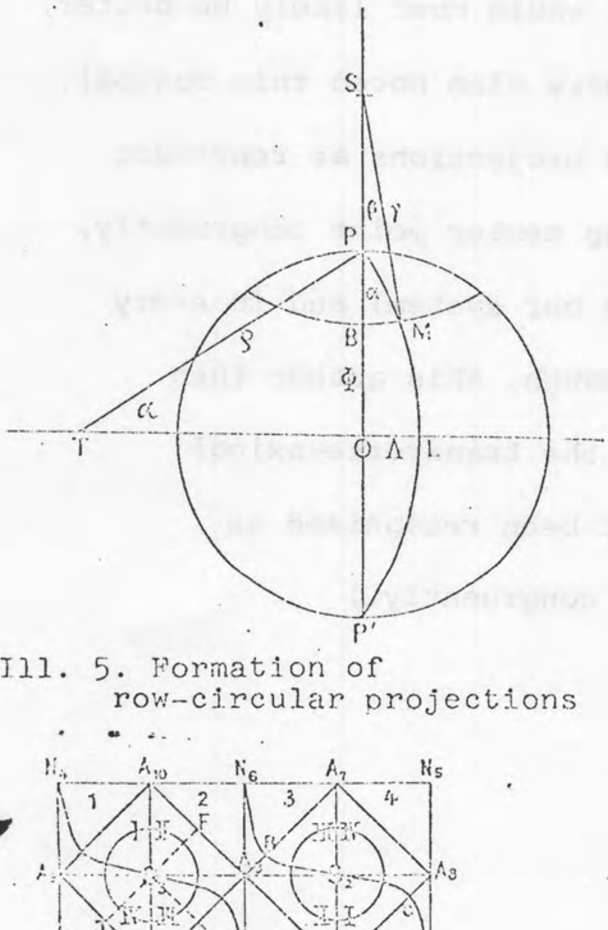
Ill. 1. Formation of conical projections



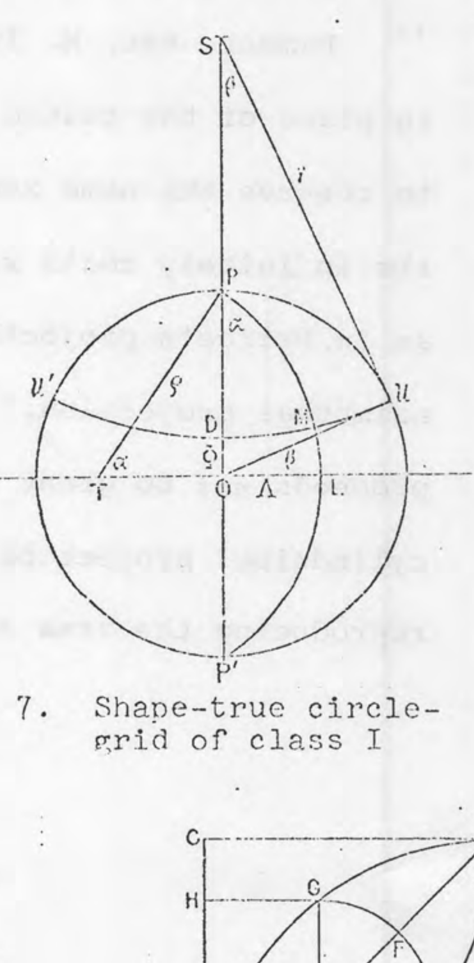
Ill. 2. Formation of plane-perspective projections



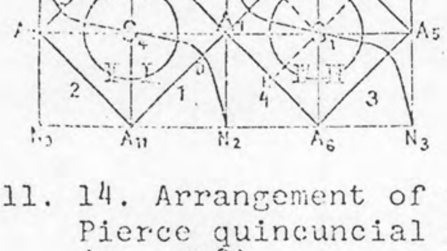
Ill. 3. Re: projection No. 3



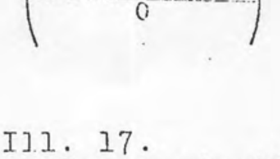
Ill. 5. Formation of row-circular projections



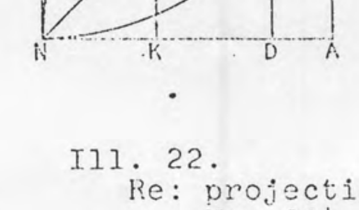
Ill. 7. Shape-true circle-grid of class I



Ill. 14. Arrangement of the Pierce quincuncial grid (No. 218)

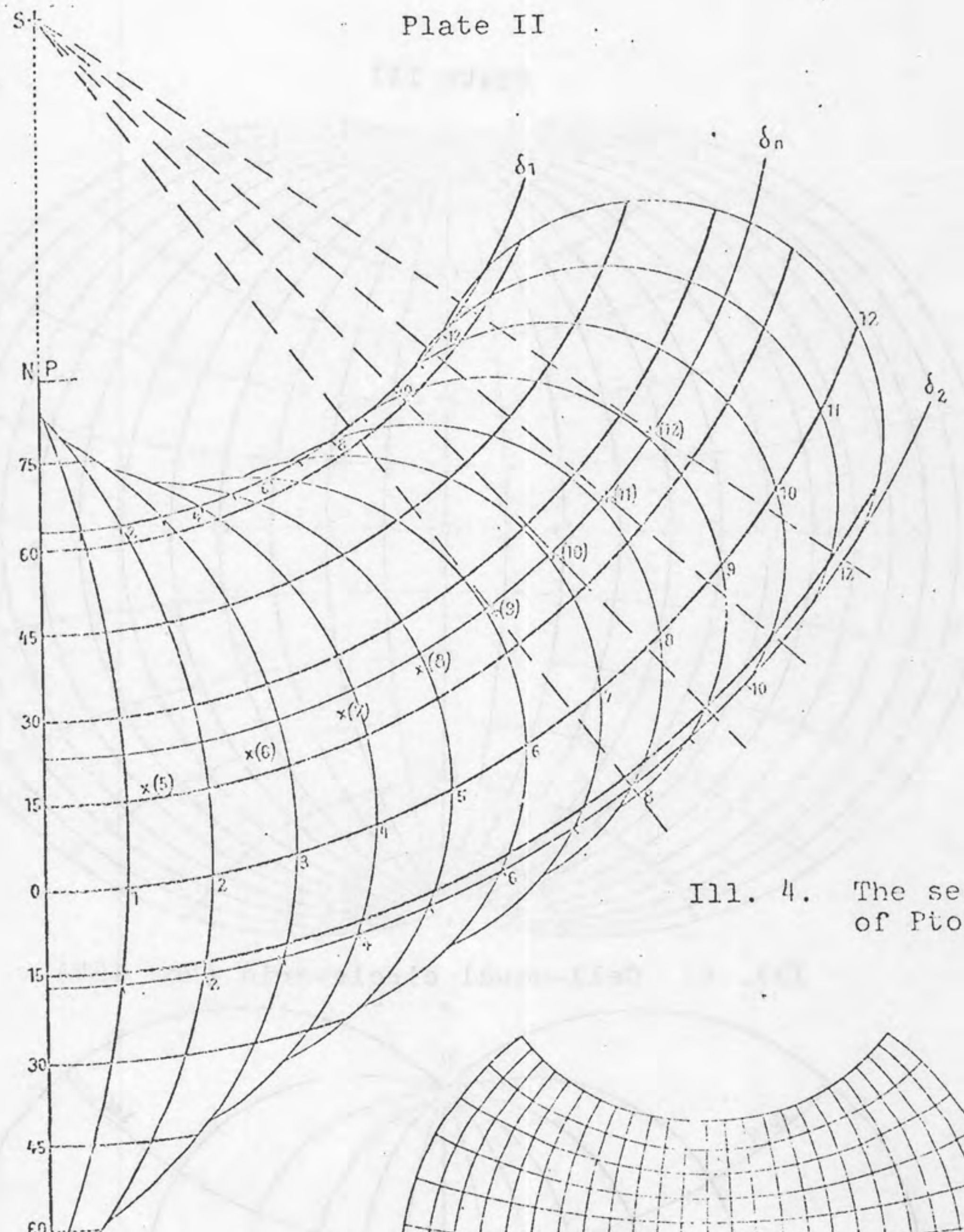


Ill. 17. Re: projection No. 227

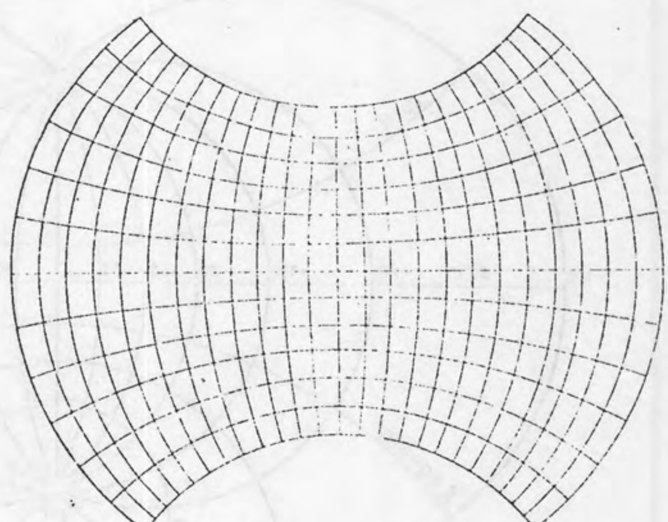


Ill. 22. Re: projection No. 224

Plate II

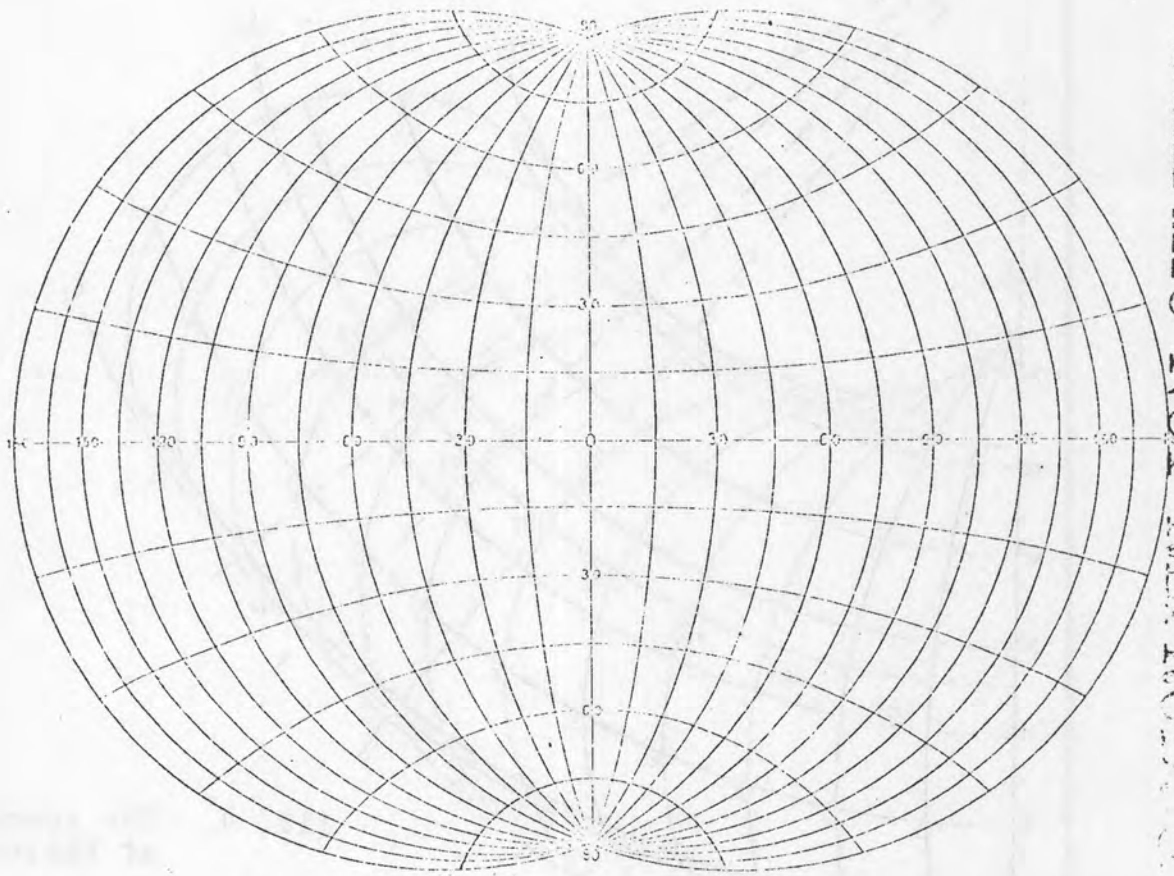


Ill. 4. The second projection of Ptolemaeus

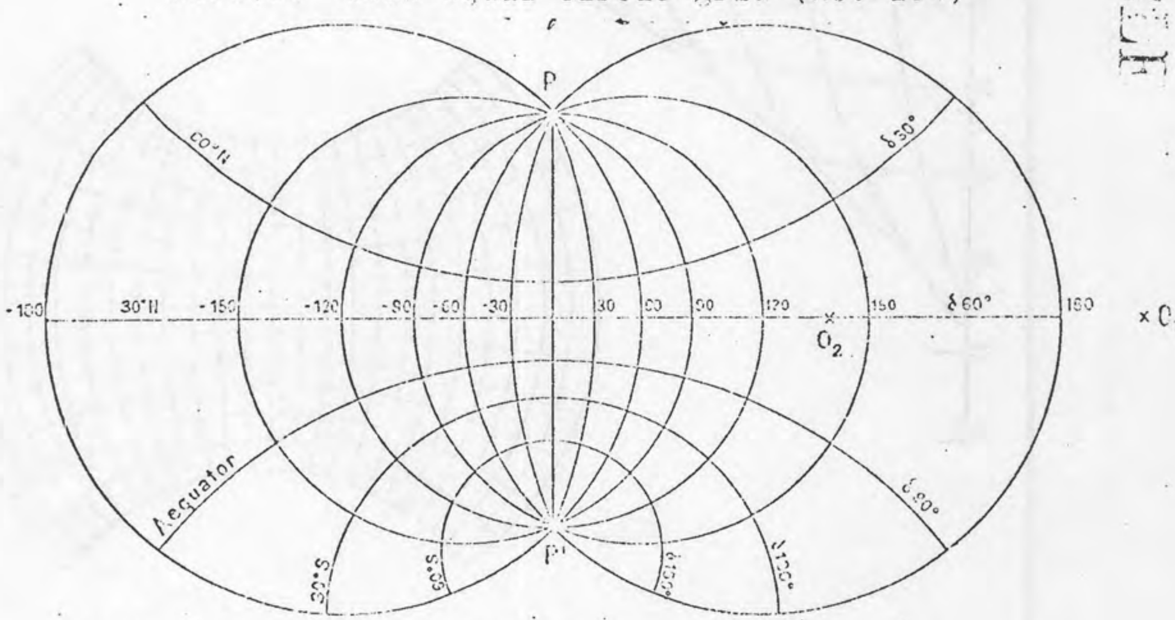


Ill. 11. Rectangular circle-grid with polar-lines (No. 180)

Plate III



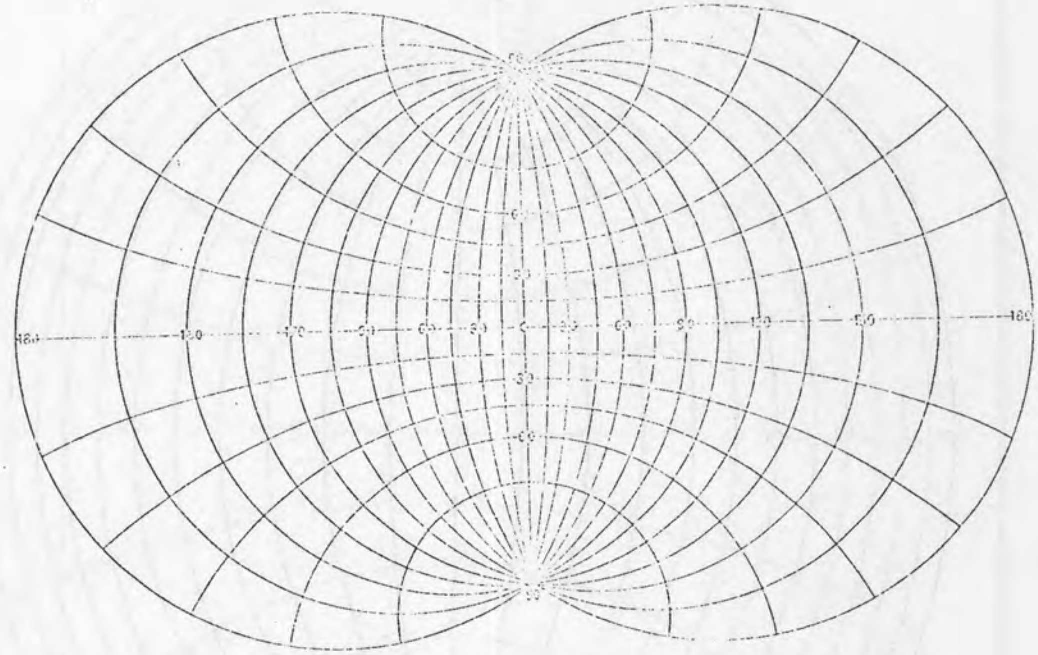
Ill. 6. Cell-equal circle-grid (No. 164)



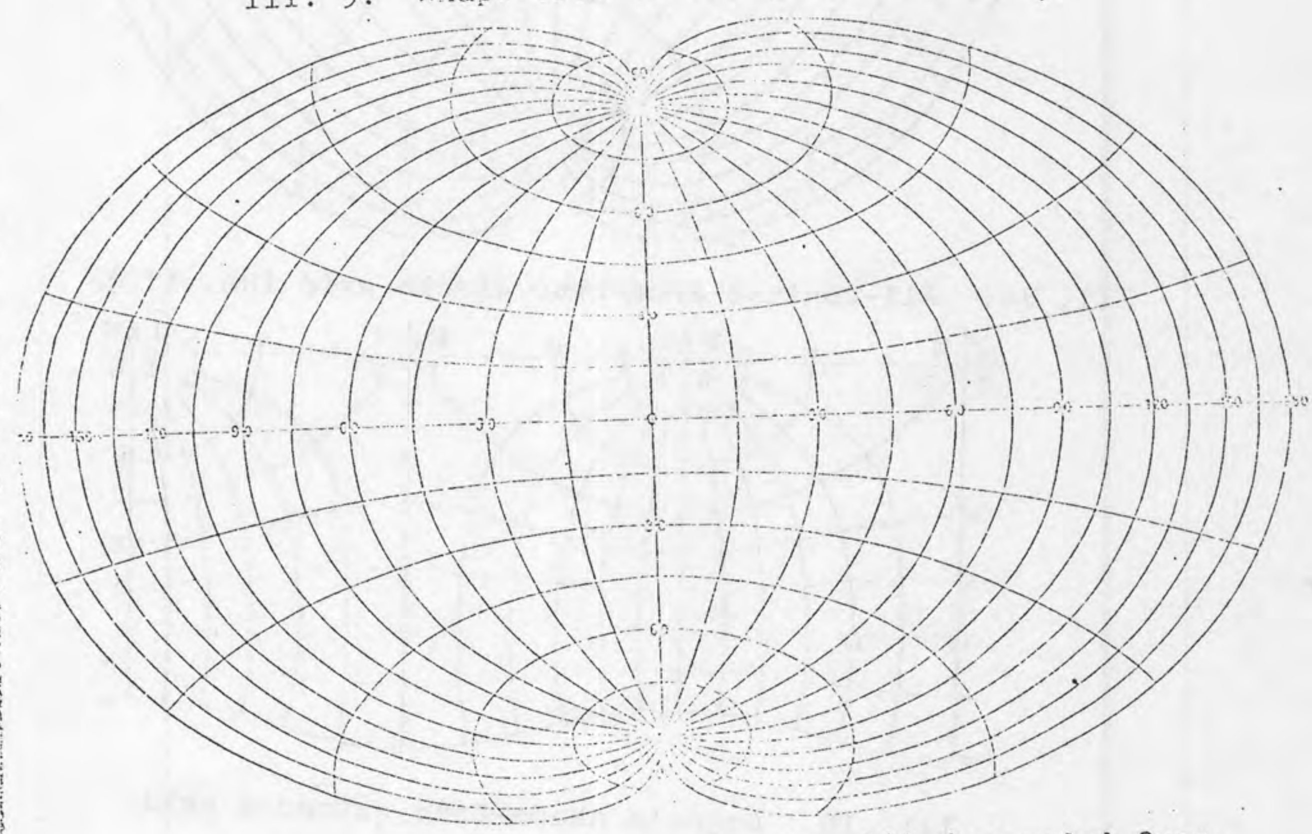
Ill. 8. Simple symmetrical shape-true circle-grid (No. 217)

(East-"Ost"
E - "O" pl.)

Plate IV

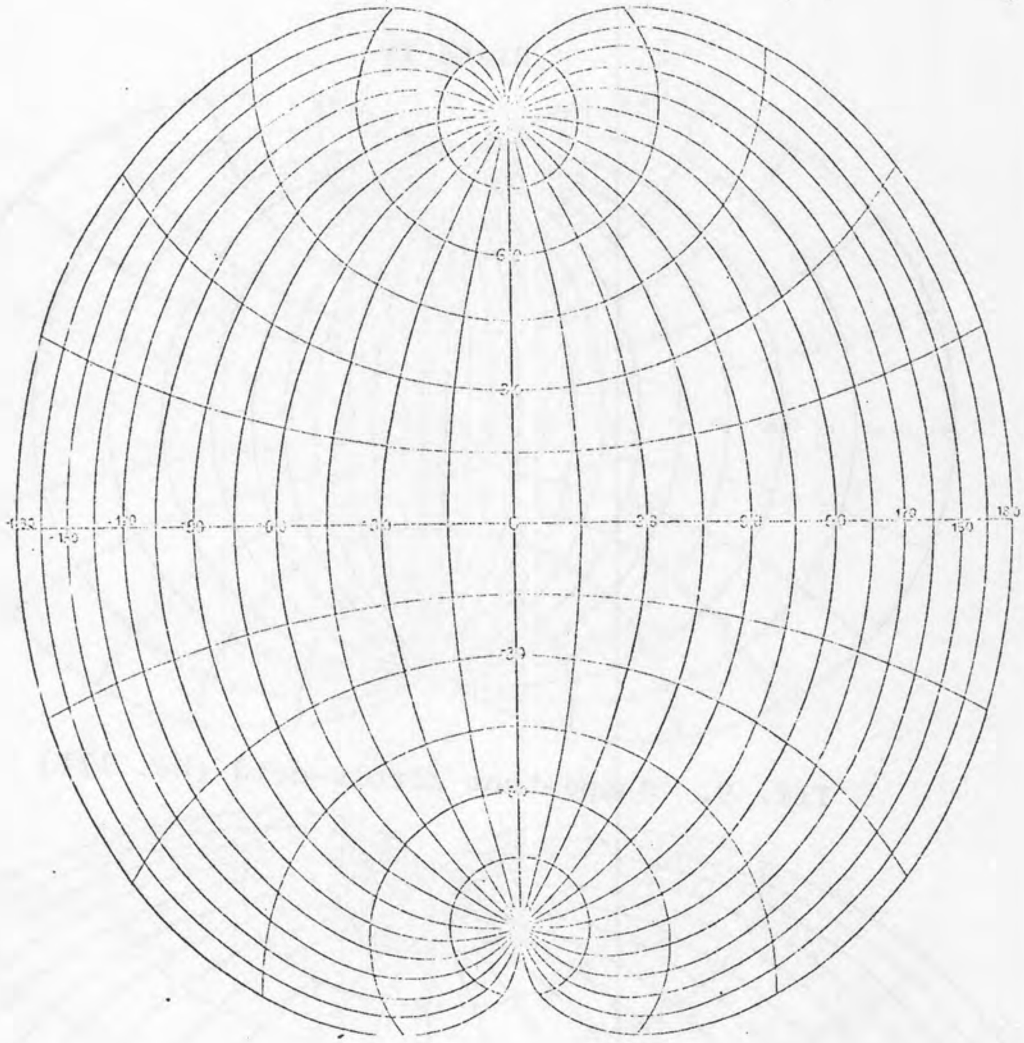


Ill. 9. Shape-true circle-grid (No. 171)

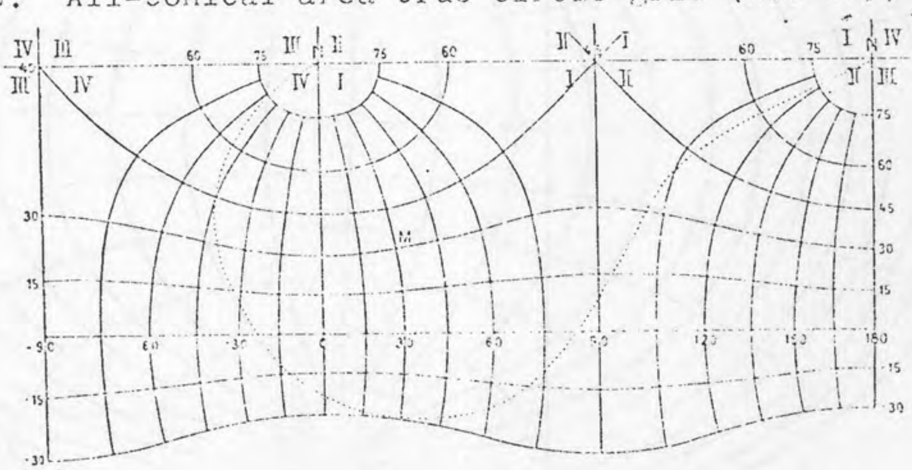


Ill. 12. Area-true projection No. 184 (expanded from circle-grid No. 179)

Plate V



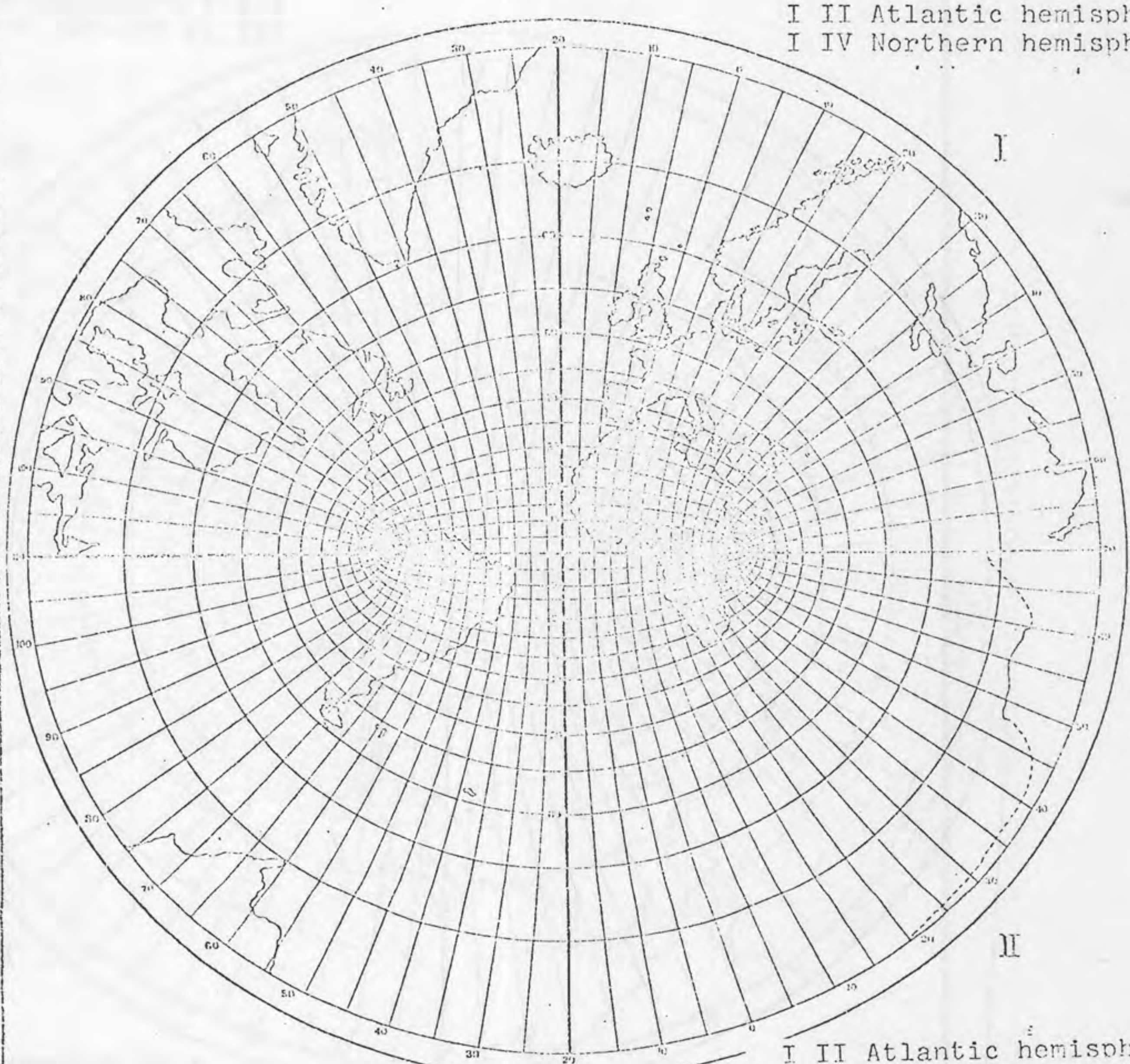
Ill. 10. All-conical area-true circle-grid (No. 179)



Ill. 15. Guyou's shape-true quinconx grid

Plate VIa

I II Atlantic hemisphere
I IV Northern hemisphere



I II Atlantic hemisphere
III II Southern hemisphere

Ill. 13. Shape-true azimuth-equal map (No. 189)

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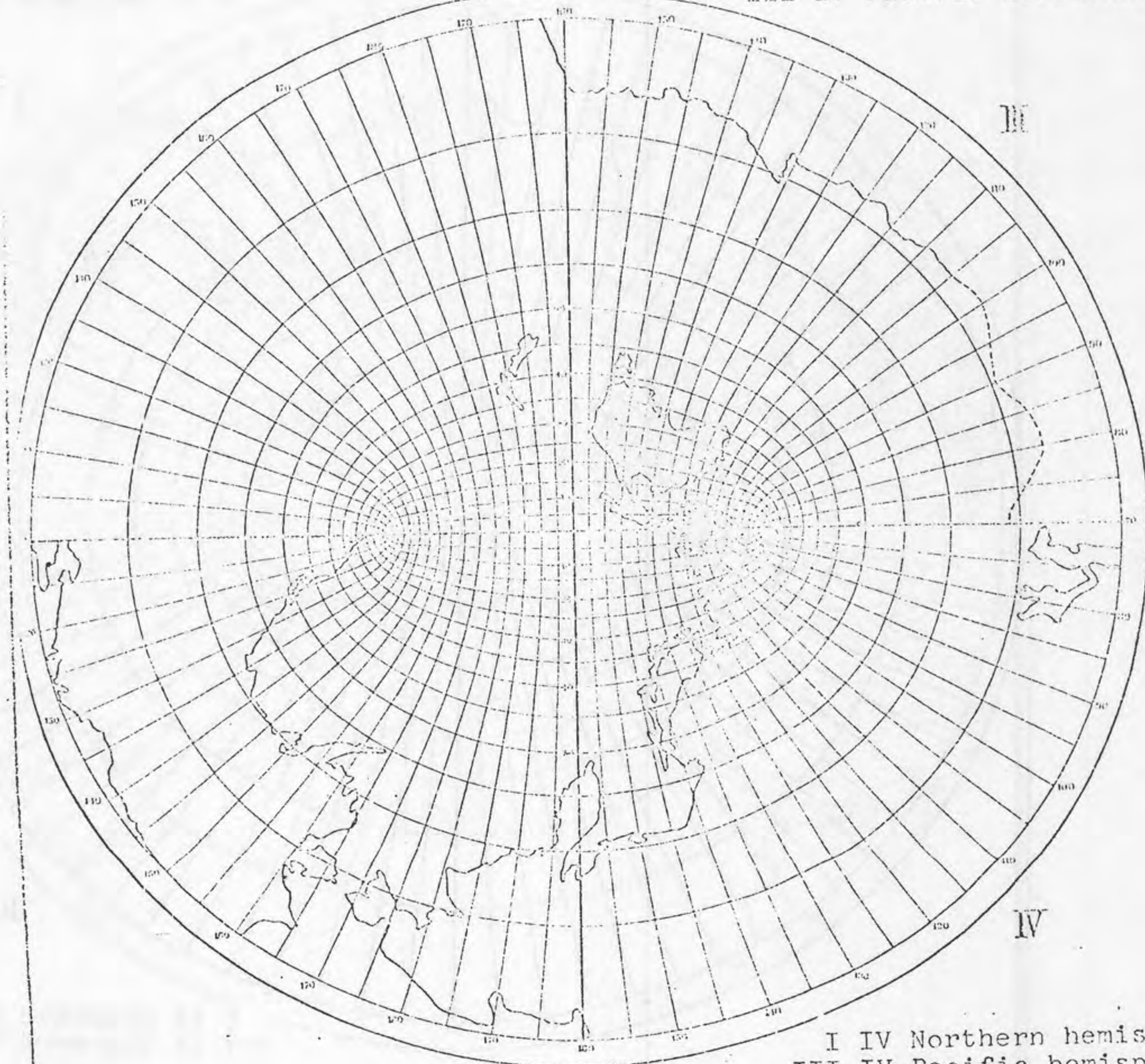
GOTHA: JUSTUS PERTHES

Plate VIb

III II Southern hemisphere
III IV Pacific hemisphere



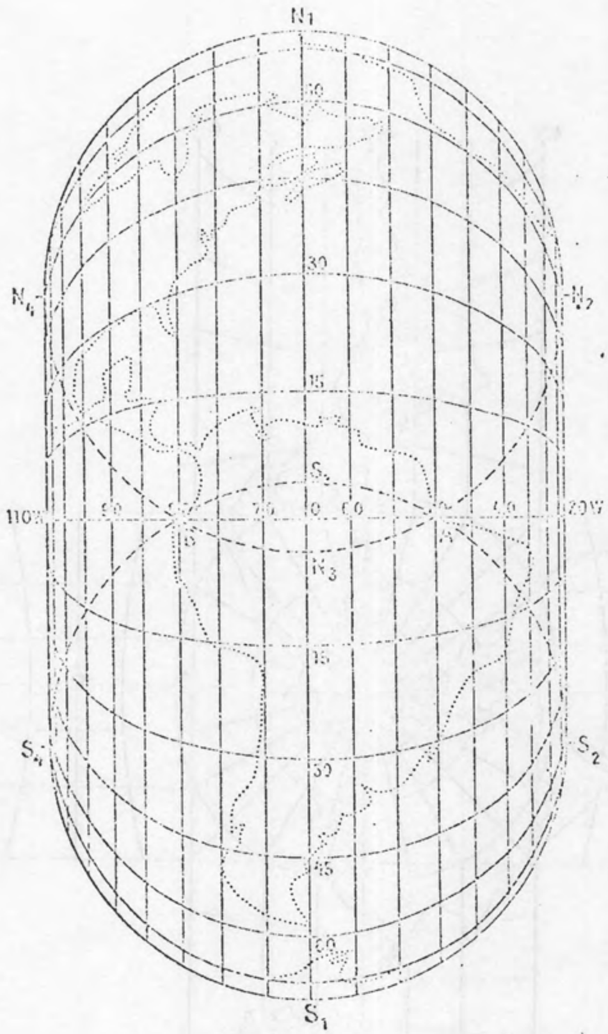
I IV Northern hemisphere
III IV Pacific hemisphere

Ill. 13. Shape-true azimuth-equal map (No. 189)

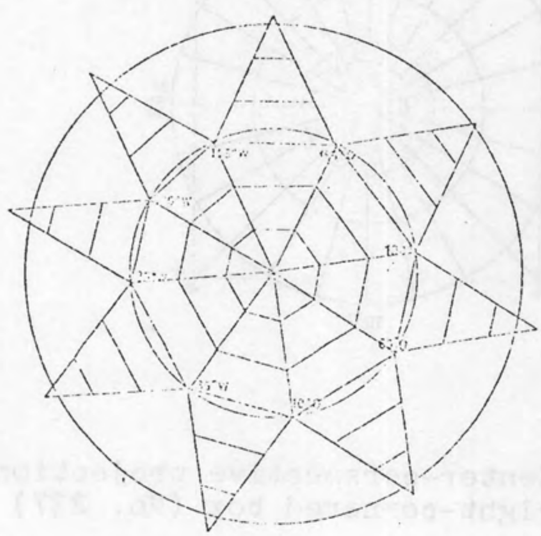
tung: Prof. Paul Langhans

GOTHA: JUSTUS PERTHES

Plate VII

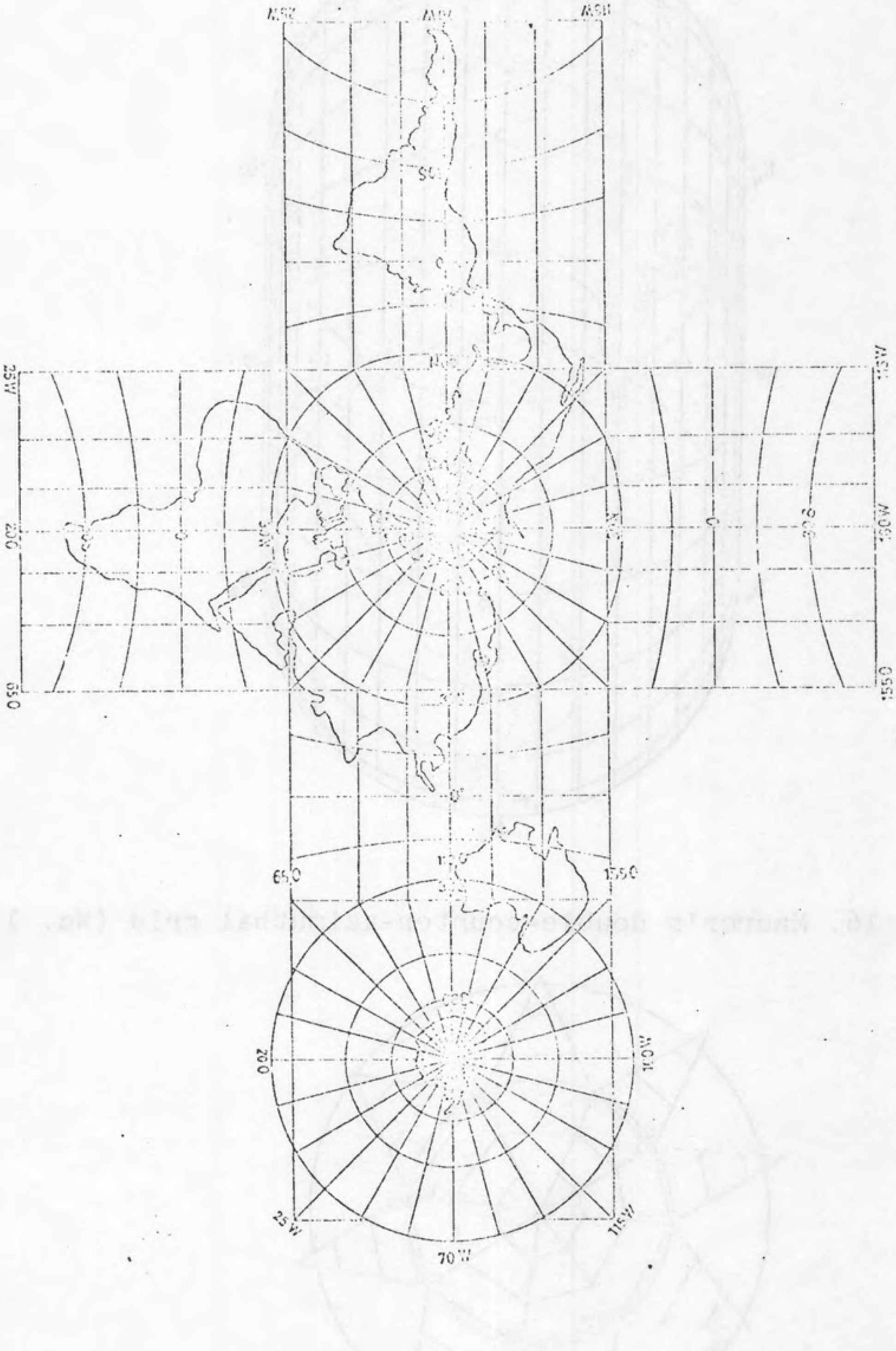


Ill. 16. Maurer's double-counter-azimuthal grid (No. 196)



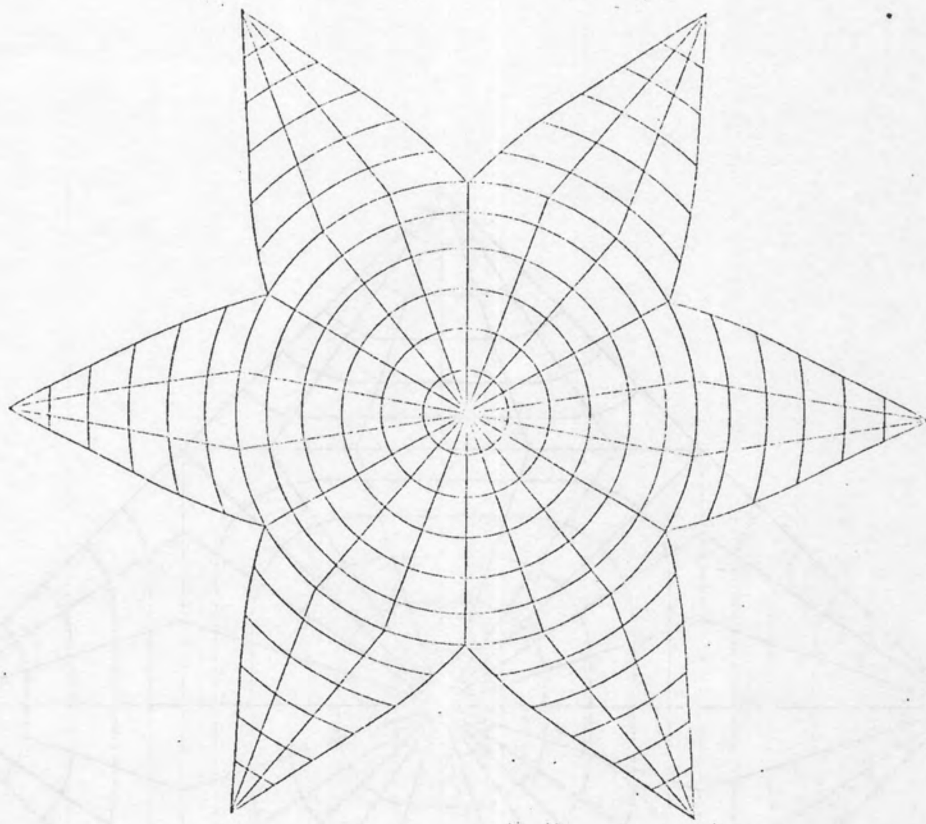
Ill. 20. Jäger's Star-projection (No. 232)

Plate VII (cont.)

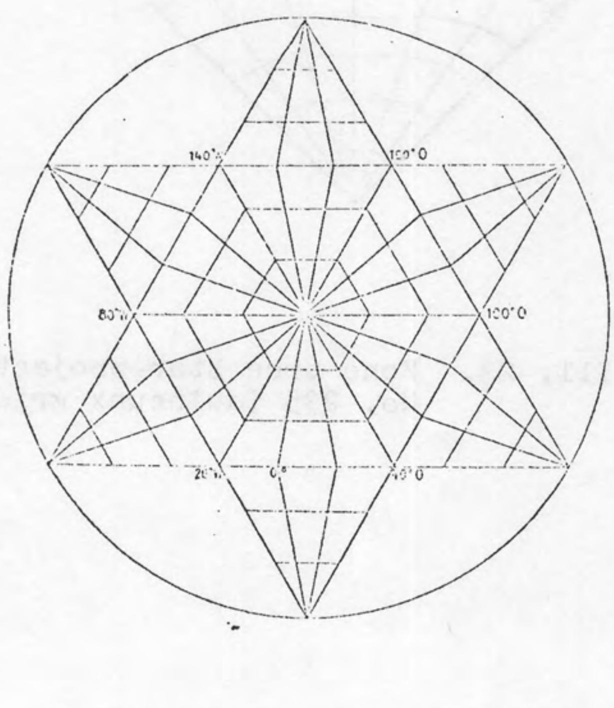


Ill. 24. Center-perspective projection onto eight-cornered box (No. 237)

Plate VIII

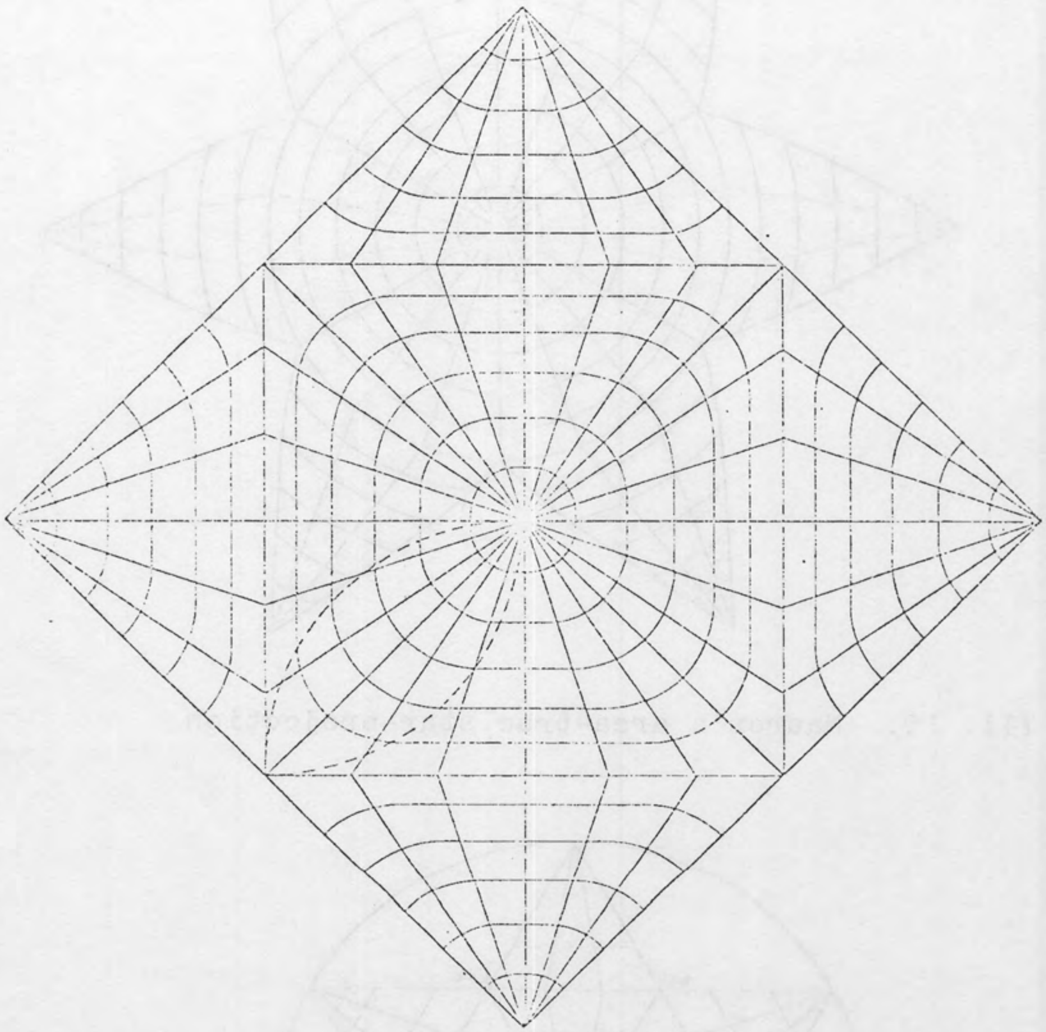


Ill. 19. Maurer's area-true star-projection



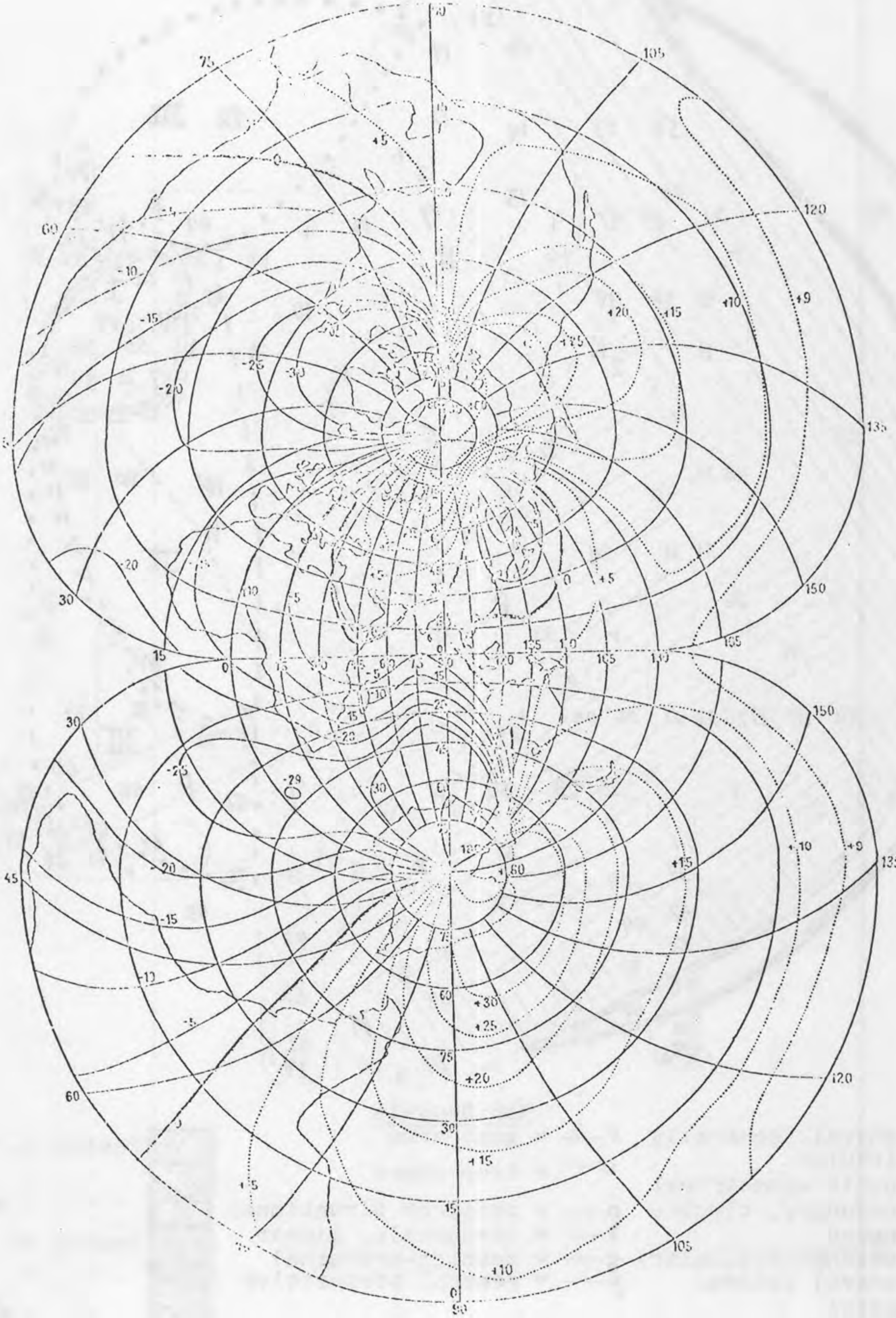
Ill. 21. Simple area-true star projection (No. 233)

Plate VIII (cont.)



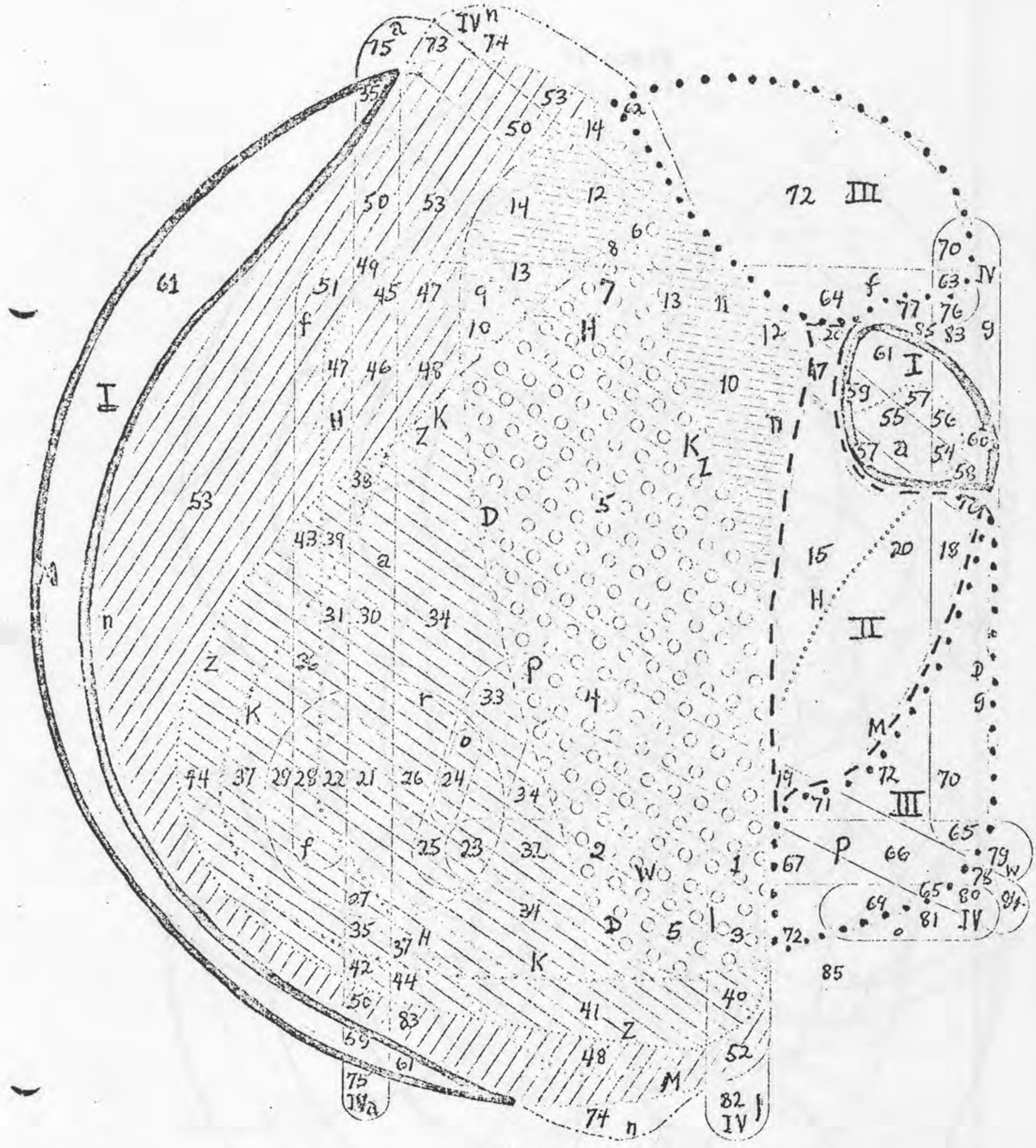
Ill. 23. Zone-true star-projection No. 235 (quincunx grid)

Plate IX



Ill. 18. Isogone Map (No. 227)

Plate X



Map Symbols

M—— = general, centrally circular
D--- = double symmetrical
n-·- = secondary, circle-spaced
z··· = zenithal (cellular)
K = general conical
r—— = radial
H··· = prime-point map
a—— = true interval

f—— = true area
w—— = true shape
o—— = straight directional
l—— = loxodromic, linear
g—— = counter-azimuthal
p—— = general perspective

Family I, Branch A
" " " B
" " L
Family II, Branch A
" " " B
" " L
Family III

System Table of Map Grids

(H = primary-line, N = secondary-line, E = projection, F = figure, TH = Tissot-Hammer, ZB = Zöppritz-Bludau, H = Herz, G = Gretschel, BF = Bourgeois-Furtwängler)

Field No.	Subbranch	Order	Class	Subclass	Group	Sort	Type	Literature
							FAMILY I: TRUE-CIRCULAR: The N are concentric circles, not straight lines. $r = f(\delta)$	
							BRANCH A: Secondary-circle-spaced. All N equal-spaced. $\alpha = g(\delta) \cdot \lambda h(\delta)$	
24, 69, 80; 22, 66, 78	A. Zenithal (cellular) All cells permutable; $h(\delta) = n$	a. straight-center (segment con...) : All H straight; $g(\delta) = 0$, $\alpha = n\lambda$	I. radial azimuthal; $n = 1$, $\alpha = \lambda$	a) Prime-point maps; $r(0°) = 0$; shape-true at point $r = \delta = 0$	A. Plane-perspective; $r = (q+c)\sin\delta : (q + \cos\delta)$; r maximum for $\delta = \Delta'$ where $\sec\Delta' = -q$, sighting point below basic circle by q, map-surface above basic circle by c	a. inner-perspective $-1 < q < 1$	center-perspective (gnomonic) (ca. 600 b.C.) $q = 0$, $c = 1$, $r = tg\,\delta$. Straight-directional (orthodromic) [Great-circle-true according to Schols $r = 2 tg\delta : (1 + \sqrt{1 + K^2 tg^2\delta})$; $K = 0$] $\Delta' = 90°$	TH 58; ZB 65 (F41), H 45; G 45; BF 256
						b. surface perspective $q = 1$	All-circular (stereographic) (Hipparch, 160-125 b.C.) $q = c = 1$, $r = 2 tg \frac{\delta}{2}$ shape-true [Great circle-true according to Schols $r = 2 tg\delta : (1 + \sqrt{1 + K^2 tg^2 \delta})$, $K = 1$] $\Delta' = 180°$	TH 59, ZB 67 (F51), H 27, G 60, BF 257
						c. exter-perspective (external) $q > 1$	distant perspective (orthographic) (Apolonius, 240 b.C.) $q = \infty$, c arbitrary, $r = \sin\delta$ $\Delta' = 90°$	TH 60, ZB 59, H 16, G 103; BF 256
							De la Hire (1704) " " ; $q = (2 + \sqrt{2})$ L; $r(90°) = 2r(45°)$ $\Delta' = 125°9$	" 65 " 71; " 91
							" " " (generalized) c arbitrary; $q = 1 + \cos\Delta/2$, $r(2\Delta) = 2r(\Delta)$ $\mathrm{cosec}\,\Delta' = 2\sin^2\frac{\Delta}{4}$	new
							Gretschel (?) 1873 " " , $q = 0{,}5 + \sqrt{1.25} = 1.618$ Length increase the same at $\delta = 0°$ and $\delta = 90°$ $\Delta' = 128°2$	H 75, G 94
25							Lowry (1825) $c = 0$, q: mean of the 18 values for which $r(\delta = n \cdot 5°) = n \cdot 18$, $n = 1, 2, 3, \ldots 18$ $\Delta' = 126°5$	G 95
71							Parent II (1704) $c = 0$, $q = \sqrt{3} = 1.732$ Greatest length increase a minimum $\Delta' = 125°3$	G 95
84							" " " $c = 1$, $q = 1$ " " " $\Delta' = 120°$	G 95
0						STIPULATIONS FOR HEMISPHERES	Parent I (herz) q from $(q+c)(c+1)\frac{dA}{dq} + (1-c)A = 0$, where $A = (q^4 - 2q^2 - 2) \cdot 2q^2(q-1)$: entire change of length increase from $\delta = 0°$ to $\delta = 90°$ a minimum $c = 1$, $q = 1.603$, $\Delta' = 128°6$	TH 61; H 74; G 94
1							Parent I (herz) as 10, except that $c = 0$; $q = 1.594$ $\Delta = 128°8$	H 75; G 94
2							Herz (1885) q from $(1+c)(q+c)\frac{dA}{dq} + 2(1-c)A(q+c)^2 : (q+1)^2 = 0$ where $A = 8q^3 + 2q^3(q-1)^2 - 1 - (q+1)^2 : q^3$, $[2K - K_0 + K_{90}]$ a minimum, where ratio of area increase at $\delta_0 = K_0$, at $\delta_{90} = K_{90}$ and its highest value K is employed for δ', where $\cos\delta' = \frac{q}{2} - \frac{3}{2q}$ { $c = 1$; $q = 2.125$; $\Delta' = 117°6$	H 76, G 96
3							Parent III (1704) (as Herz above) $c = 0$, $q = 2.100$; $\Delta' = 118°4$	H 77; G 96
4							James (1857) $q = 1.5$; k_1 and k_2 greatest and least ratio of change of length at each point. c so determined that $\int[(k_1 - 1)^2 + (k_2 - 1)^2]$ becomes minimum above the surface of the hemisphere $c = 0.5772$; $\Delta' = 131°8$	H 209, G 96
5							Clark (1862) - Fiorini (1881) The same minimum stipulation for hemisphere as James No. 14, but q and c are variable $q = 1.470$; $c = 0.548$; $\Delta' = 132°9$	H 210
6						STIPULATIONS FOR CALOTTES	Escher (ca. 1850) $r(\Delta) = \Delta_1$; $\sec\Delta' = q$; $0 \sin\Delta_1 - q(\Delta_1 - \sin\Delta_1) = \Delta_1 \cos\Delta_1$, $c = 1$; $q = (\sin\Delta_1 - \Delta_1\cos\Delta) : (-\sin\Delta)$	TH 63
7							" , generalized $r(\Delta_1) = \Delta_1$; $r(\Delta_2) = \Delta_2$; Examples $\Delta_1 = 90°$; $\Delta_2 = 45°$, $q = 1.67$; $c = 0.454$; $\Delta' = 126°.8$ or $\Delta_1 = 135°$; $\Delta_2 = 67°5$; $q = 1.382$, $c = 0.868$, $\Delta' = 136°.3$	new
8							Hammer (1887) calotte area = picture area. $(q+c)\cos\Delta/2 = q + \cos\Delta$ $c = 1$; $q = (\cos\Delta/2 - \cos\Delta) : (1 - \cos\frac{\Delta}{2})$	TH 64
9							" generalized for 2 calottes. $q = 1 + 2\cos\frac{\Delta_1}{2}\cos\frac{\Delta_2}{2}$; $c = 2[\cos\frac{\Delta_1}{2} + \cos\frac{\Delta_2}{2}] - q$. Examples: $\Delta_1 = 90°$, $\Delta_2 = 45°$, $q = 2.307$; $c = 0.455$, $\Delta' = 115°.7$	new
0							Tissot I (1878-80) Length increase at $\delta = 0$ so great as at Δ; c arbitrary; $q = 0.5(1 + \sqrt{5 + 4\cos\Delta})$. Example: $\Delta = 90°$; $q = 1.618$; $\Delta' = 128°.2$	TH 61
1							Tissot II " " Highest value of ratio of area increase K in calotte Δ a minimum. c arbitrary q from $q^3 \cdot q^2 - (2 + 3\cos\Delta)q \cdot \cos^2\Delta - \cos\Delta = 1$. Example: $\Delta = 90°$; $q = 2.148$; $\Delta' = 117°.7$	TH 62

No	FNo.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
								FAMILY I (cont.): true-circular. Branch A - Secondary-circle spaced	
22	26					B. Not perspective	a. doubly projected	Lidman (1877) First from globe center onto contiguous cone at $\delta = 45°$; then parallel to the earth's axis projected onto equator plane. $r = 1 - \frac{1}{\sqrt{2}} tg(45° - \delta)$; $r(90°) = \sqrt{2}$, $r(135°) = \infty$	TH 74
23	22						b. area-true	Lambert (1772) (isospheric, isemeric, named incorrectly after Lorgna); $r = 2 \sin \frac{\delta}{2}$; $r(180°) = 2$	TH 65, ZB61 (F67), H176, G241; BF262
24	21						c. intervening	Mercator (1569) (central-interval-true, named incorrectly after Postel (1581); $r = \delta$; $r(180°) = \pi$	TH 72, ZB56 (F116), H96; G241; BF264
25	26							Breusing (1892) $r = 2\sqrt{tg \frac{\delta}{2} \sin \frac{\delta}{2}}$ = geometric mean between shape-true FNo 2 and area-true FNo. 23. $r(180°) = \infty$	TH 73, ZB 79, H XIII, BF265
26								Airy (1861) Balance of errors. Integral of $[(\frac{k_1}{k_2} - 1)^2 + (k_1 k_2 - 1)^2]$ a minimum over hemisphere surface. $r = 2 \cot \frac{\delta}{2}$ lognat $\sec \frac{\delta}{2} + tg \frac{\delta}{2}$; $r(90°) = 1.693$	TH 71; H221, G247, BF265
27								James and Clarke (1862) Integral of $[(k_1 - 1)^2 + (k_2 - 1)^2]$ a minimum over calotte Δ; $r = 2 \cot \frac{\delta}{2}$ lognat $\sec \frac{\delta}{2} + 2 tg \frac{\delta}{2} \cot^2 \frac{\Delta}{2}$ lognat $\sec \frac{\Delta}{2}$, $\Delta = \delta = 90°$ yields $r(90°) = 1.386$	H220; G248
28	28				b) ring maps $r(0°) \gtrless 0$		a. area-true	Hammer (1887) $r(0°) > 0$; $r = \sqrt{r_0^2 + 4 \sin^2 \frac{\delta}{2}}$. All N too great, no N scale-true. $r(90°) > \sqrt{2}$, e.g., $r(0°) = 0.5$; $r(90°) = 1.5$	TH 69
29							"	Maurer (1914) $r(0°) < 0$, $r = \sqrt{\sin^2 \delta_m + 2 \cos \delta_m - 2 \cos \delta}$. δ_m scale-true. Map applies only for $\delta > \Delta$, where $2 \cos \Delta = \sin^2 \delta_m + 2 \cos \delta_m$, e.g., $\delta_m = 90°$, $\Delta = 60°$, $r(90°) = 1$	Pet. M.H. 1914, II, p.64
30	27						b. circle-interval-true	" $r(0°) < 0$, $r = \sin \delta_m - \delta_m + \delta$. δ_m scale-true. Map applies only for $\delta > \Delta$, where $\Delta = \delta_m - \sin \delta_m$; e.g., $\delta_m = 90°$, $\Delta = 32°.7$; $r(90°) = 1$	new
31	29						c. neither area nor interval-true	$r = a + \sin \frac{\delta}{2}$	"
32	33			II. Conical $0 < n < 1$ $r(\delta_m) \sin \gamma = \delta_m$ δ_m scale-true γ = half cone angle	a) prime-point maps $r_0 = F(0) = 0$	A. cone-perspective $r = (q + c) \sin \delta : [q + \gamma \sin(\gamma + \delta)]$ sighting pt. below basic circle by q, above basic circle by c	a. intersecting conical $\delta_m \gtrless (90° - \gamma)$	General case. No special stipulations concerning q, c.	TH 99
33								Center-perspective. $q = 0$, $r = c \sin \delta : \sin(\gamma + \delta)$; δ_m from $\sin(\gamma + \delta_m) = c \sin \gamma$	TH 101
34								Center-perspective and zone-equal. Murdoch II (1759). Area-equal zone between δ_1 and δ_2. $c = \sec \frac{\delta_1 + \delta_2}{2} \sqrt{\cos \frac{\delta_2 - \delta_1}{2}}$; $r = \sqrt{\cos \frac{\delta_2 - \delta_1}{2}} [tg \frac{\delta_1 + \delta_2}{2} + tg(\delta - \frac{\delta_1 + \delta_2}{2})]$; $q = 0$; $\gamma = 90° - \frac{\delta_1 + \delta_2}{2}$	TH 102, H87; G144
35							b. tangential-cylindrical $\delta_m = 90° - \gamma$	General case $c = \operatorname{cosec} \gamma$, q and γ arbitrary. Special case of 32	TH 99
36								Center-perspective $q = 0$, $r = \sin \delta \operatorname{cosec} \gamma \operatorname{cosec}(\gamma - \delta)$	TH 103
37								Surface-perspective, Braun's stereographic cone projection (1868) $q = 1$, $c = 2$; $n = 0.5$; $\gamma = 30°$; $r = 3 \sin \delta : [\sin 30° + \sin(30° + \delta)]$	TH 100; H90
38	34					B. zone-perspective	intersecting conical	$r = (c - \cos \delta) \sec \gamma$, $\delta_m = 180° - 2\gamma$ scale-true. "rolled off cone" after Tissot-Hammer. Special case ($\delta_2 = 0$ and $\delta_m = \delta_1$) of 45	TH 98
39	32					C. shape-true	a. tangential-cylindrical	Lambert-Gauß (1772 and 1822) $r = tg\, \delta_m \left[\frac{tg \frac{\delta}{2}}{tg \frac{\delta_m}{2}}\right]^n$; $n = \cos \delta_m$; $\gamma = 90° - \delta_m$. Secondary-circle δ_m scale-true	TH 77; ZB133, G153, BF273, H109
40							b. intersecting conical	" " $r = \frac{\sin \delta_1}{n} \left[\frac{tg \frac{\delta}{2}}{tg \frac{\delta_1}{2}}\right]^n$; $n = \frac{\lg \sin \delta_1 - \lg \sin \delta_2}{\lg tg \frac{\delta_1}{2} - \lg tg \frac{\delta_2}{2}}$; secondary circles δ_1 and δ_2 scale-true. (Harding's sky map)	TH 81; ZB140 (F85), H112, G153
41	30					D. center-interval-true $r = \delta$	intersecting conical	n arbitrary. Calotte Δ area-equal for $\Delta \sqrt{n} = 2 \sin \frac{\Delta}{2}$; δ_m scale-true for $n = \sin \delta_m : \delta_m$. Example: $\Delta = 90°$; $n = 0.8125$, $\gamma = 54°.3$; $n \cdot 360° = 292°.5$	
42								$n = 0.739$, $\gamma = 47°38'$, special case of 41, in which $\Delta = \delta_m$; $n \cdot 360° = 266°$; $\Delta = \delta_m = 74°52'$; $c = 1.142$	
43								$n = 0.5$ Schjerning I (1882) $\delta_m = 108°36'$; $\gamma = 30°$; $\Delta = 159°27'$	ZB 187 (F119)
44						E. area-true	intersecting conical	Lambert (1772) $r = 2 \sin \frac{\delta}{2} : \sqrt{n}$; scale-true and shape-true δ_m, where $\cos^2 \frac{\delta_m}{2} = n$. Equal shape-distortion at δ_1 and δ_2, where $n = \cos \frac{\delta_1}{2} \cos \frac{\delta_2}{2}$. Examples: $\delta_2 = 0$; $\delta_1 = 90°$; $\delta_m = 65°32'$; $n \cdot 360° = 254°.45$; $n = 1 : \sqrt{2}$	TH 84-87; ZB128 (F79) G151; BF271
45	37				b) ring-maps $f(0°) > 0$	A. zone-perspective	a. intersecting cone	δ_1 and δ_2 scale-true, $n = \cos \frac{\delta_1 + \delta_2}{2}$; $c = \cos \frac{\delta_1 - \delta_2}{2} : \cos \frac{\delta_1 + \delta_2}{2}$; $r = (c - \cos \delta) : \sin \frac{\delta_1 + \delta_2}{2}$; $\gamma = 90° - \frac{\delta_1 + \delta_2}{2}$	TH 98
46						"	b. tangential-cylindrical	δ_1 scale-true; $n = \cos \delta_1$; $c = \sec \delta_1$; $r = (\sec \delta_1 - \cos \delta) : \sin \delta_1$; $\gamma = 90° - \delta_1$. Special case $\delta_2 = \delta_1$ of 45	

No.	F.No.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
'7 '8 '9	35					B. circle-interval-true $r = r(\delta_m) + \delta - \delta_m$	a. tangential-conical b. intersecting cone	Ptolemaeus I (130 A.D.) simple cone · E. $r = tg\,\delta_m + \delta$; $n = \cos\delta_m = \sin\gamma$; δ_m scale-true De l'Isle (1745) δ_1 and δ_2 scale-true. $n = \cos\frac{\delta_1+\delta_2}{2}\sin\frac{\delta_1-\delta_2}{2} : \frac{\delta_1-\delta_2}{2}$; $r = \frac{1}{n}\sin\frac{\delta_1+\delta_2}{2}\cos\frac{\delta_1-\delta_2}{2} - \frac{\delta_1+\delta_2}{2} + \delta$. Greatest shape-distortion at δ_3, where $\cos\delta_3 = n = \sin\gamma$ Example: $\delta_1 = 70°$, $\delta_2 = 20°$, $\gamma = 43°14'$, $n \cdot 360° = 246°6$; $\delta_3 = 46°46'$ Murdoch I (1754) Ring area between δ_1 and δ_2 area-equal. $n = \cos\frac{\delta_1+\delta_2}{2}$; $r = tg\frac{\delta_1+\delta_2}{2}\sin\frac{\delta_1-\delta_2}{2} : \frac{\delta_1-\delta_2}{2} - \frac{\delta_1+\delta_2}{2} + \delta$; $c = \sec\frac{\delta_1+\delta_2}{2}\sin\frac{\delta_1-\delta_2}{2} : \frac{\delta_1-\delta_2}{2}$; $n \cdot r(\delta_m) = \sin\delta_m$	TH 94; ZB 112 H 92, G 136, BI 274 TH 95; ZB 115 (F 79), H 101, G 135, BI 275 TH 96; H 87; G 142
'0	37					C. circle-interval-similar $r = r_0 + m\delta$	intersecting cone	Murdoch II " Ring area between δ_1 and δ_2 area-equal. $n = \cos\frac{\delta_1+\delta_2}{2}$; $r = (tg\frac{\delta_1+\delta_2}{2}\cot\frac{\delta_1-\delta_2}{2})\frac{\delta_1-\delta_2}{2} + m(\delta - \frac{\delta_1+\delta_2}{2})$, $m = \sin\frac{\delta_1-\delta_2}{2}\,tg\frac{\delta_1+\delta_2}{2}(\frac{\delta_1-\delta_2}{2})^2$, $c = \sec\frac{\delta_1+\delta_2}{2}\cot\frac{\delta_1-\delta_2}{2} \cdot \frac{\delta_1-\delta_2}{2}$	TH 97; G 147
'1 '2	36					D. area-true $r = \sqrt{r_0^2 + \frac{4}{n}\sin^2\frac{\delta}{2}}$	a. tangential cone b. intersecting cone	With scale-true N $n = \cos\delta_m$; $r_m = tg\,\delta_m$, $r_0 = \sec\delta_m - 1$; $r = \sqrt{r_0^2 + \frac{4}{n}\sin^2\frac{\delta}{2}}$. Special case $\delta_2 = \delta_1$ of 52 Equal length- and shape-distortion at δ' and δ'', where $\cos\delta' + \cos\delta'' = 2\cos\delta_m$ Albers 1805 with 2 scale-true N δ_1 and δ_2. $n = \cos\frac{\delta_1-\delta_2}{2} \cdot \cos\frac{\delta_1+\delta_2}{2}$; $r_0 = \frac{2}{n}\sin\frac{\delta_1}{2}\sin\frac{\delta_2}{2}$; $r = \sqrt{r_0^2 + \frac{4}{n}\sin^2\frac{\delta}{2}}$. Greatest length shortening at δ_3, where $\cos\delta_1 + \cos\delta_2 = 2\cos\delta_3 = 2n$	TH 73, ZB 130, H 131, G 145, BI 271
53 54 55	39 38 41		b curved-cellular $g(\delta)$ not constant	radial-circular $\alpha = g(\delta) + \lambda$	All H are arcs of radius R	A. Prime-point maps $f(0) = 0$	a. All H scale-true, R = 1 b. $R > \frac{\pi}{2}$	Wiechel (1879) area-true $\alpha = \lambda + \frac{\delta}{2}$; $r = 2\sin\frac{\delta}{2}$ interval-true $\alpha = \lambda + \beta$, where $\beta = \delta : 2R$; $r = \delta$ not interval-true, e.g., $\alpha = \lambda + \beta$, where $\sin\beta = \sin\frac{\delta}{2} : R$, $r = 2\sin\frac{\delta}{2}$	TH 70, H 166 new "
56 57 58	42 43 44					B. Ring-maps $f(0) = r_0 > 0$	a. interval-true b. area-true c. neither interval nor area-true	$\alpha = \lambda + \beta$, where $\sin\beta = \frac{r_0+\delta}{2R}$; $2R > r_0 + \pi$; $r = r_0 + \delta$ $\alpha = \lambda + \beta$, where $\sin\beta = r : 2R$; $2R > \sqrt{r_0^2 + 4}$; $r = \sqrt{r_0^2 + 4\sin^2\frac{\delta}{2}}$ $\alpha = \lambda + \beta$, where $\sin\beta = r : 2R$, $2R > r_0 + 2$; $r = r_0 + 2\sin\frac{\delta}{2}$	" " " "
59 60	49 50	B not cellular $h(\delta)$ not constant	a. conical-circular (cutrue-conical)	I. r equal-spaced $r = m(tg\,\delta_m + \delta - \delta_m)$	a) all N scale-similar	A. point-ring-maps $f(0) > 0$	a. circle-interval-true $m = 1$	Mercator-Bonne (1584, 1752), $n = 1$. All N scale-true. Area-true. Shape-true on central-H and on N δ_m; $n \gtrless 1$. All N and all areas changed in ratio n. Nowhere shape-true. At $n > 1$ the map overlaps the tangential-cone of δ_m in zone, where $n\sin\delta - \delta > tg\,\delta_m - \delta_m$	TH 106, ZB 142 (F 87) H 135; G 159; BF 275
61 62	51 53		Central H = straight line $\alpha = 0$ for $\lambda = 0$ $g(\delta) = 0$	interval-similar	$r\alpha = n\lambda\sin\delta$ n = constant		b. circle-interval similar $m \gtrless 1$	$mn = 1$. Area-true. No N scale-true. Nowhere shape-true. $n = 1$. All H scale-true. Area-similar at ratio m.	
63 64	45 46					B. prime point-maps $f(0) = 0$ central-interval-similar	a. center-interval true $m = 1$, $r = \delta$	$n = 1$. Stab (1514), also named after Werner. Special case $\delta_m = 0$ of 59. All N scale-true. Area-true. Shape-true on central-H. Also Schjerning's map no. IV $n < 1$. Example: $n = 0.5$. Schjerning I; $\alpha = \frac{\lambda}{\delta}\sin\delta$; area- and distance-similar, nowhere shape-true. Special case $\delta_m = 0$ of 60	TH 107; ZB 144, 198 (F 124), H 174; G 162; BF 279 ZB 200 (F 127)
65 66	47 48					$r = m\delta$	b. central-interval-similar $m \gtrless 1$	$mn = 1$. Area-true, Special Case of 61. Example: Schjerning V with another uniform globe $m = \sqrt{2}$, $n = 1 : \sqrt{2}$; $n = 1 . \sqrt{2}$; $r = \delta\sqrt{2}$, $\alpha = \lambda\sin\delta : 2\delta$ $n = 1$. Distance-true " " " 1.2. " " " " " " " $m = 2$ $r = 2\delta$; $\alpha = \lambda\sin\delta : 2\delta$	" " " " " "
67 68 69	46	[The combination of 68 and 69 as one map belongs in Family I.]			b) length μr of N etc. to world map border $\mu r = f(\delta)$	Prime-point maps $r_0 = 0$	central-interval-true $m = 1$, $\delta_m = 0$ $r = \delta$	Schjerning III (1904) World-map border 2 full circles which are contiguous at $\delta = 0$. $r = \delta$; $\alpha = \frac{\lambda}{\pi}\arccos\frac{\delta}{\pi}$ for $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$. Scale-true N $\delta_m = 79°27'$. True-shape at points $\lambda = 0$ or $180°$, $\delta = 0°$, $180°$ or $79°27'$ Schjerning II (1904) World-map border ellipses with equator and central-H as axes. Left half of map. $r = \delta$, $\alpha = \lambda$ for $\delta < \frac{\pi}{2}$; $\alpha = 2\beta\lambda : \pi$ for $\delta > \frac{\pi}{2}$, where $tg^2\beta = (5\pi^2 - 4\delta^2) : (4\delta^2 - \pi^2)$ Schjerning II (1904) Right half. World-border of graphically determined arcs from the poles to equator-point $r = \pi$; $\alpha = 0$; $\alpha = \lambda$ for $\delta < \frac{\pi}{2}$; $\alpha = 2\beta\lambda : \pi$ for $\delta > \frac{\pi}{2}$, where β is graphically determined function of δ	ZB 195 (F 123) ZB 193 (F 121) " " " "

No.	Fig.	Sub branch	Order	Class	Sub class	Group	Sort	Type	Literature
0, 1	51			II not equal-spaced	a) Length of N acc. to world-map border $\mu = f(d)$	a Area-true $\int_0^r \mu d r = 4\pi \sin^2 \frac{d}{2}$	a line-ring maps $r_0 > 0$	Nell's E. μ = center from 51 and 59. $r\mu = \pi[r\cos\delta_m + \sin(r - tg\,\delta_m + \delta_m)]$. $N\delta_m$ scale-true; r_0 from $r_0^2\cos^2\delta_m - \cos(r_0 - tg\,\delta_m + \delta_m) = 1 - 4\cos\delta_m + \cos^2\delta_m$ " " Example of 70: $\delta_m = 45°$; $\cos\delta = 0.5303 - r^2 : \sqrt{32} + 0.5\cos(r° - 12°17'.75)$; $\mu° = 150° : \sqrt{2} + \frac{180°}{r} \sin(r° - 12°17'.75)$; $\alpha = \mu\lambda : (2\pi)$; $r_0 = 0.3709$; $r_{180} = 2.5775$. Scale-true and shape-true at δ_m.	Pet. Mitt. 1890, p. 99
2, 3	47						b. prime-point-maps $r_0 = 0$	" " Examples of 70. $\delta_m = 0$; $\cos\delta = \cos^2\frac{\mu}{2} - \frac{r^2}{4}$; $\mu = 150°[1 + \frac{\sin r}{r}]$ = center from 23 and 63. $\alpha = \mu\lambda : (2\pi)$; $r_{180} = 2.1971$ Maurer. World map border of 3 semi-circles. $\cos\delta = \frac{2}{3\pi}(\mu\cos\mu - \sin\mu) - \frac{1}{3}$; $r = 4\sin\frac{\mu}{2} : \sqrt{3}$; $x = \mu\lambda : (2\pi)$; $r_{180} = 2.3094$. Scale-true $\delta_m = 45°46'$. Shape-true at $\lambda = 0°$ or $180°$, $\delta = 0°$ or $45°46'$	new

FAMILY I : TRUE-CIRCULAR

BRANCH B: not secondary-circle-spaced

No.	Fig.	Sub branch	Order	Class	Sub class	Group	Sort	Type	Literature
4	55	A. Radial-circular		I. All H straight		A. prime-point maps	a. interval-true	$r = \delta$, $\alpha = \pi\sin\frac{\lambda}{2}$. True-shape on curves $2\sin r = (\pi^2 - \alpha^2)$	Pet. Mitt. 1914, II, p.64, No. XI
5	57	$\alpha := \pi$ for $\lambda = \pi$		$\alpha = g(\lambda)$		$r_0 = 0$	b. not interval-true	$r = \sin\frac{\delta}{2}$; $\alpha : \pi = \sin\frac{\lambda}{2}$. True-shape on curves $r = \sqrt{1 - \frac{\pi}{\alpha}\cos\frac{\lambda}{2}}$ or $\pi\cos\frac{\lambda}{2} = 2\cos^2\frac{\delta}{2}$	
6	61	for not $\alpha = \lambda$				B Line-ring maps	" "	$r = a + \sin\frac{\delta}{2}$; $\alpha = \pi\sin\frac{\lambda}{2}$ $\pi\cos\frac{\alpha}{2} = (\sin\delta\cos\frac{\delta}{2}) : (a + \sin\frac{\delta}{2})$	
7	55			II. The H are curves		prime-point-maps	a interval-true	Maurer, 1919. Double-interval-true from points $\varphi = 0$; $\lambda = \pm L$. $y = (d_1^2 - d_2^2) : 4L$; $x = \sqrt{\frac{d_1^2 + d_2^2}{2} - L^2 - y^2}$, where $\cos d_1 = \cos\varphi\cos(L + \lambda)$; $\cos d_2 = \cos\varphi\cos(L - \lambda)$ Field No. applies for grid of points $\varphi = 0$; $\lambda = \pm 1$.	Ann. d. Hyd. 1919, p. 77; Deetz and Adams, p. 164
8	54						b center-azimuthal direction-point	Hammer Interval-true $r = d$; $y = -d\sin A$; $x = d\cos A$, where $\cos d = \sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos\lambda$ and $\cot A = tg\,\varphi\cos\varphi_0\,cosec\,\lambda - \sin\varphi_0\cot\lambda$ True-shape and field no. 54 apply for direction-point $\lambda = 0$, $\varphi = \varphi_0$. Field no. 82 has the polar grid, field no. 69 for $\varphi_0 = Q$	Pet. Mitt. 1910, p. 153; ZB 220
9, 0	56						$(\lambda = 0; \varphi = \varphi_0)$	r, as in all-circular E no. 2, not true-shape. $r = \frac{1}{2} tg\frac{d}{2}$; $y = x\,tg\,A$; $x^2 = r^2 - y^2$. d and A as no. 78. Field no. 56 applies for grid of the direction-point. Polar grid as no. 78. r as in area-true E no. 23, zone-true, not area-true. $r = 2\sin\frac{d}{2}$, $y = x\,tg\,A$; $x^2 = r^2 - y^2$. d and A as no. 78. Field no. 56 applies for grid of the direction-point. Polar grid as no. 78.	ZB 221 (F151)
1	59	B. conical-circular		Ring maps		Interval-true	All H are straight lines	$2H$, d, and d_2, scale-true; $d_1 = d_m + \mathcal{E}$; $d_2 = d_m - \mathcal{E}$ Simplified cone E after De l'Isle (1745). (incorrectly called Mercator's E); $r_1 = f(d_1)$; $r_2 = f(d_2)$, $\alpha_1 = \lambda\sin d_1 : r_1$ at Nd_1	ZB 120, BF 275 footnote
2		Highest value 2π				$f(d) : tg\,d_m\,f\,d_m$	All H are arcs	Ptolemaeus II (130 A.D.) $d_1 = 27°$, $d_2 = 66°10'$; $d_3 = 106°30'$ $\alpha_2 = \lambda\sin d_2 : r_2$ at Nd_2	H 187; G 158

No	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature

FAMILY II: STRAIGHT-SYMMETRICAL.

All N are parallel straight lines. The Grid is symmetrical to the central-H as well as to the base-line. $x=F(\varphi)=-F(\varphi)$; $y(\lambda/\varphi)=-y(-\lambda/\varphi)$

BRANCH A: secondary-circle-spaced : $y=\lambda \cdot h(\varphi)$

No	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
33	8	A. straight-cellular $h(\varphi)=n=R$ (R = radius of the cylinder) All H are parallel straight lines	I. cylindrical (true cylindrical projections)	a) line-ring maps		A. circle perspective-true $x=\sin\varphi\frac{q+R}{q+\cos\varphi}$ $y=R\lambda=n\lambda$	a. contiguous-cylindrical $R=n=1$	$q=1$, $a=2$. Braun's "stereographic-cylinder E" (1867) $N(\varphi=0)$ scale-true and shape-true. $x=tg\frac{\varphi}{2}$; $y=\lambda$. 1 N shape-true.	TH 31 H91
34			— — — —	$F(\pm 90°)$ finite				$q=0.4$; $a=1.4$ " "modified Mercator E" (1867). $N(\varphi=0)$ scale- and shape-true; $\varphi_m=68°40'.6$ shape-true. $x=1.4\sin\varphi:(0.4+\cos\varphi)$; $y=\lambda$; 2 N shape-true.	TH 31
35			cylinder-circular $x=F(\varphi)$ and cylinder-radial $y=n\lambda$			$R=q:(a-1)$	b. intersecting-cylindrical $R=n=\cos\varphi_m$	$q=1$, $a=(1+\cos\varphi_m):\cos\varphi_m$ et stereographic intersecting cylinder E. $N(\pm\varphi_m)$ scale- and shape-true. $x=(1+\cos\varphi_m)tg\frac{\varphi}{2}$; $y=\lambda\cos\varphi_m$. 3 N shape-true.	New
36						— — — —	$n<1$ c. free cylindrical $n>1$	$q=0.4$; $R=n=1.04$, $a=1.3846$. Maurer's circle-perspective E. 4 N shape-true $\varphi_m=\pm 22°57'.5$ and $\varphi_m=\pm 67°4'.1$. $x=1.44\sin\varphi:(0.4+\cos\varphi)$; $y=1.04\lambda$. 4 N shape-true	"
37	7			— —		B. area true $x=\frac{1}{n}\sin\varphi$; $y=n\lambda$	a. contiguous-cylindrical $n=1$	Lambert (1772), zone perspective $x=\sin\varphi=\lambda$ $N(\varphi=0)$ scale- and shape-true. Special case $\varphi_m=0$ of 38.	TH 27; ZB 159 (F97); H 154; G 113; BF 267
38							b. intersecting-cylindrical	$x=\sin\varphi\sec\varphi_m$; $y=\lambda\cos\varphi_m$. $N(\pm\varphi_m)$ scale- and shape-true. Equal shape-distortion at φ_1 and φ_2, where $\cos\varphi_1\cos\varphi_2=\cos^2\varphi_m$	TH 26; ZB 160 BF 267
39				— —			$n=\cos\varphi_m$	Behrmann (1910) $\varphi_m=30°$; $x=2\sin\varphi:\sqrt{3}$; $y=\lambda\sqrt{3}:2$. Least average uppermost shape-distortion on the world map.	Pet. Mitt. 1910, II, plate 27
c	6			— —		C. circle intersection-true	a. contiguous cylindrical $n=1$	Quadratic flat map, Thorne (1527) $x=\varphi$, $y=\lambda$. $N(\varphi=0)$ shape-true	TH 28, ZB 155 (F95); H 95; G 111, BF 269
11						$x=\varphi$	b. intersecting cylindrical $n<1$	Rectangular " " Marinus of Tyrus (100 A.D.) $x=\varphi$, $y=\lambda\cos\varphi_m$; $N(\pm\varphi_m)$ shape-true. $n=\cos\varphi_m$	TH 28, ZB 157; H 106 G 112; BF 269
12[x]	5	x) In No 12, FNo 5 and family II apply only for $q=0$; family IV and FNo 33 (SNo 197) apply for $q\gtrless 0$.		b) prime-point-maps		A. cylinder-perspective	a. intersecting-cylindrical $n=\cos\varphi_m$	q arbitrary. $x=\cos\varphi_m[\sin\varphi+q(1-\cos\varphi)]\sec\varphi$; $y=\lambda\cos\varphi_m$; $N(\pm\varphi_m)$ scale-true, $N(\varphi=0)$ shape-true. FNs 5 and fam. II for $q=0$. Family IV, SNo 197, FNo 33 apply for $q\gtrless 0$.	TH 29
13	5		— —	$F(\pm 90°)=\infty$		Sighting point q below basic-circle	b. contiguous-cylindrical $n=1$	$q=0$. Wetch center-perspective. $x=tg\varphi$; $y=\lambda$. $N(\varphi=0)$ scale- and shape-true. Special case ($\varphi_m=0$) of 93.	TH 30; H 124; G 131
14	1					B. shape-true not perspective	course-linear (loxodromes)	Etzlaub (1511) - Mercator (1569) (Mercator projection or navigation-map). Contiguous-cylindrical $x=\text{lognat}\ tg(45°+\frac{\varphi}{2})$; $y=\lambda$. " " $N(\varphi=0)$ scale-true. Loxodromes = straight lines	TH 24; ZB 163 (F99); H 114; G 119; BF 267
15								" " " " intersecting-cylindrical $x=\cos\varphi\ \text{lognat}\ tg(45°+\frac{\varphi}{2})$; $y=\lambda\cos\varphi_m$ $N(\pm\varphi_m)$ scale-true. Loxodromes = straight lines	
16	3		— —	— — — —		C. not shape-true	a. course linear	Intersecting-cylindrical, Etzlaub-Mercator affine transformed $x=m\ \text{lognat}\ tg(45°+\frac{\varphi}{2})$; $y=n\lambda$, $m\gtrless n$	new
17	5					not perspective	b. not course-linear	" " $x=\sqrt[3]{tg\varphi}$; $y=n\lambda$. $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m=n$	new
18	9	B. not cellular (not true cylindrical) H are curves	a. prime-point-maps $F(\pm 90°)=\infty$	I. Area-true $y=\lambda\cos\varphi\frac{dx}{d\varphi}$		$x=n\,tg\varphi$; $y=\frac{\lambda}{n}\cos^3\varphi$		Type according to value of n; $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m=\sqrt{n}$. Shape-true points $\lambda=0$; $\varphi=\pm\varphi_m$; H = algebraic curves $y(n^2+x^2)^{3/2}=n^2\lambda$ with asymptote $y=0$	New
19	10			II. All N scale-true		$x=n\,tg\varphi$; $y=\lambda\cos\varphi$		" " " " " . All N scale-true. Shape-true points $\lambda=0$, $\varphi=\pm\varphi_m$, where $\cos\varphi_m=\sqrt{n}$. H = algebraic curves $y(n^2+x^2)^{1/2}=n\lambda$ with asymptote $y=0$	"
50	11	D. Ring maps	I. central H equal-spaced $x=m\varphi$	c) pointing-maps with 2 poles		A. All N are scale similar $y=n\lambda\cos\varphi$	a. interval-true $m=1$	$n=1$ Mercator's (1606) area-true E (usually named after Sanson (1650) and Flamsted (1700)). True-shape at central-H and base-line. Earth-axical. Distance-true	TH 43; ZB 164 (F104); H 159; G 160; BF 276
01	12	$F(\pm 90°)$ finite						$n\gtrless 1$ area- and distance-similar, nowhere shape-true. Least shape-distortion on hemisphere for $2m^2=\sqrt{4+\pi^2}$	
52	13		interval-similar	$y=0$ for $\varphi=\pm 90°$		area-similar	b. interval-similar	$mn=1$ area-true. No N scale-true; nowhere true-shape	TH 42
03	14					H = sine lines	$m\gtrless 1$	$n=1$. All N scale-true. Area-similar, nowhere true-shape.	

SNo	FNo	Subbranch	Order	Class	Subclass	Group	Sort	Type	Literature
104	14	See footnote p.		re. No. 104-106		B. not all N scale-similar but	a. Each H a pair of straight lines each thru a pole and dividing	General case (m arbitrary). $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = m(\pi-2\varphi_m):\pi$. Shape-true at $\varphi=\lambda=0$. Area-equal globe-biangles, if $m=2:\sqrt{\pi}$	
105	12					on base-line	point on base-line $y=m\lambda(\pi-2\varphi):\pi$	Special case $m=1$, Interval-true. Base-line and central-H scale-true. Shape-true on base-line $x=0$.	new
106	14					(x=0) $y=m\lambda$		Special case $m=2:\sqrt{\pi}$. Area-equal globe-biangles between the H, $N(\varphi_m=\pm12°.0)$. Shape-true on base-line $x=0$	new
107	14						b. Each H a semi-ellipse through the 2 poles and dividing point	General case (m arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m = \frac{m}{\pi}\sqrt{\pi^2-4\varphi_m^2}$. Shape true on base line $x=0$. Area-equal globe-biangles, if $m=\sqrt{8}:\pi$	new
108	12						on base-line $y=\frac{m\lambda}{\pi}\sqrt{\pi^2-4\varphi^2}$	Special case $m=1$. Interval-true. Base-line and central-H scale-true. Shape-true on base-line $x=0$. Apianus II (1524), used by Arago	
109	14							Special case $m=\sqrt{8}:\pi$. Area-equal globe-biangles between the H; $N(\varphi_m=\pm33°.2)$ scale-true. Shape-true on base-line $x=0$	TH 52; Wagner
110	14				b) line-ring maps	A. combination maps between No. 90	y from No. 100-103 for $n=m$	General case (m arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m=m:(2-m)$. Shape-true on base-line. Area-equal globe-biangles, if $m=2:\sqrt{\pi+2}$	
111	12					($x=m\varphi$; $\varphi=m$) and sub-class	$y=\frac{m\lambda}{2}(1+\cos\varphi)$ H² sine-lines	Special case $m=1$. Winkel I (1922) Interval-true. Scale-true and shape-true on base-line $x=0$	Geog. Ztschr. 1922, p.177
112	14					100 to 109		Special case $m=2:\sqrt{\pi+2}$. Eckert V. Area-equal globe-biangles betw. the H. $N(\varphi_m=\pm37°.4)$ scale-true. Shape true on base line.	ZB 212 (F142)
113	14	"	"	" re. No. 113-115		$y=\frac12(m\lambda+y_1)$	y_1 from No. 104-106, $y=m\lambda(\pi-\varphi):\pi$	General case (m arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m=m(\pi-\varphi):\pi$. Shape true at points $\lambda=0$, $\varphi=0$ and $\varphi=\pm37°.83$. Area-true globe-biangles, if $m=\sqrt{8:(3\pi)}$	
114	12					Not more than 3 scale-true	H=straight-line-pairs	Special case $m=1$ Eckert I Interval-true. Scale-true: central-H and 3 N, $\varphi=0$, $\varphi=\pm37°.83$. Shape true as No. 96	ZB 203 (F130)
115	14							Special-case $m=\sqrt{8:3\pi}$. Area-equal globe-biangles betw. the H. Scale-true $N(\varphi_m=\pm47°.2)$ " " " "	
116	14					$N(\pm\varphi_m$ and $\varphi=0)$	y_1 from No. 107-109	General case (m arbitrary) $N(\pm\varphi_m)$ scale true, where $\cos\varphi_m=\frac{m}{2\pi}\sqrt{\pi^2-4\varphi^2}$. Shape-true on base line. Area-equal globe-biangle, if $m=4:\sqrt{\pi(4+\pi)}$	
117	12						$y=\frac{m\lambda}{2}\left[1+\frac{\sqrt{\pi^2-4\varphi^2}}{\pi}\right]$	Special case. Interval-true. Scale- and shape-true on base-line.	
118	14						H=semi-ellipses	Special case $m=4:\sqrt{\pi(4+\pi)}$ Eckert III. Area-equal globe-biangles betw. the H. Scale true $N(\varphi_m=\pm36°.0)$. Shape true as 116.	ZB 206 (F134)
119	14					B. Combination maps betw. No. 91	y_1 from No. 100-103	General case (m and n arbitrary) $N(\pm\varphi_m)$ scale-true, where $\cos\varphi_m=\frac{2}{n}\cos\varphi'$. Shape-true at $\lambda=0$, $\varphi=\varphi_w$, where $\cos\varphi_w=(n\cos\varphi'):(2m-n)$. Area true globe biangles, if $mn=4(2+\pi\cos\varphi')$	
120	12					$x=m\varphi$; $y=n\lambda\cos\varphi'$	$x=m\varphi$; $y=n\lambda\cos\varphi'$	Special Case $m=n=1$ Winkel (1922) Interval-true. Scale-true $N(\pm\varphi')$. Shape-true at $\lambda=0$; $\varphi=\pm\varphi'$	
121	14					and subclass 100 to 109	$y=\frac{n\lambda}{2}(\cos\varphi'+\cos\varphi)$	Special case $m=n=2:\sqrt{2+\pi\cos\varphi'}$ Area-true globe-biangles betw. the H. Scale-true $N(\pm\varphi_m)$, where $\sec\varphi_m=\sqrt{2+\pi\cos\varphi'}-1$; Shape-true at $\lambda=0$, $\varphi_w=\pm\varphi'$	Geog Ztschr. 1922, p.178
122	12					$y=\frac12(n\lambda\cos\varphi'+y_1)$ At most 2 scale-true N		Special case, No 120 and 121 combined. $\cos\varphi'=\frac{2}{\pi}$: $\varphi'=50°27'.6$. Interval-true, and area equal globe-biangles. Scale-true central-H and $N(\varphi_m=\pm50°27'.6)$. Shape-true at $\lambda=0$; $\varphi_w=\pm50°27'.6$	
123	14							Special case of 119. $n=1$; $m=4:(2+\pi\cos\varphi')$; $\varphi'=\pm59°43'.1$; $m=0.753$; area-equal globe biangles. Equal shortening of Central-H and base-line. Scale-true $\varphi_m=\pm\varphi'$. Shape-true at $\lambda=0$; $\varphi=\pm65°50'.14$.	Work of K.H. Wagner, p.24
124	14					C. with 4 scale-true N ($\pm\varphi_1$ and $\pm\varphi_2$)	every H a pair of straight lines through dividing points of scale-true N of a hemisphere	General case $x=m\varphi$; $y=\frac{\lambda}{\varphi_2-\varphi_1}[(\varphi_2\mp\varphi)\cos\varphi_1+(\varphi_1\mp\varphi)\cos\varphi_2]$	
125	12							Special case $m=1$. E by Donis (1482) and Donny (1849) $x=\varphi$; y (as 124), interval true	
126	14							Special case $m=\frac{2}{\pi}(\varphi_2-\varphi_1):[(45°-\varphi_1)\cos\varphi_2-(45°-\varphi_2)\cos\varphi_1]$. Globe-biangles area-equal. Example: $\varphi_1=30°$; $\varphi_2=60°$; $x=1.864\varphi$; $y=\lambda(1.732-0.699\varphi)$	TH 45 H 108
127	14					D. N-scale acc. to equation	$c=1$	General case (m arbitrary). Scale-true $N(\pm\varphi_m)$, where $\cos\varphi_m=n\cos\frac23\varphi_m$. Shape-true at $\lambda=0$; $\varphi=\pm\varphi_m$	
128	12					$y=n\lambda\cos^c p\varphi$	$p=2/3$	Special case $m=1$ K.H. Wagner Interval true. Wagner's example: $\varphi_m=\pm40°$; $n=0.887$.	
129	14					c = whole no. $0<p<1$	$y=n\lambda\cos\frac23\varphi$	Special case $mn=\frac23\csc\frac{\pi}{3}=0.7698$. Globe-biangles area-equal	K.H. Wagner, p.17

No	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
130	12	Cf. footnote						Special case of 129. $m=1$, $n=0.7696$. Globe-biangles area-equal and interval-true E. Scale-true central H and $N(\varphi_m = \pm 49°57')$	
131		P re No 132		Central H not equal-spaced	a) line true F; $\pi\sin\varphi = \int_0^{x} y_\pi \frac{dx}{d\varphi} d\varphi$	A. lengths of N acc. to prescribed world map border	a. F(x) acc to 110-119 Line-ring maps	F(x) acc to 112, H: sine-line Eckert VI (1906) $m = 2:\sqrt{\pi+2}$; x from $\sin\varphi = \frac{\pi}{2}(x + m\sin\frac{x}{m})$, $y = \frac{m\lambda}{\pi}(1+\cos\frac{x}{m})$. Scale-true on $N(\varphi_m = \pm 49°11')$ Shape-true at $\lambda=0$; $\varphi = \pm 49°11'$	ZB 214
132								F(x) acc to 115, H-pairs of straight-lines. Eckert II (1906) $m = \sqrt{8}:3\pi$; $x = m\pi - \sqrt{m^2\pi^2 - 2\pi\sin\varphi}$; $y = \lambda(m - \frac{x}{\pi})$. Scale true $N(\varphi_m = \pm 55°10')$ Shape-true at $\lambda=0$; $\varphi = \pm 55°10'$	" 204 BF 279
133				$x = f(\varphi)$ not im...	$y_\pi = y$ for $\lambda = \pi$			F(x) acc to 118, H: semi-ellipses. Eckert IV (1906) $m = 4:\sqrt{\pi(4+\pi)}$; $x = \frac{m\pi}{2}\sin\sigma$; $y = \frac{m\lambda}{2}(1+\cos\sigma)$, where $(4+\pi)\sin\varphi = 4\sin\sigma + \sin\sigma + 2\sigma$. Scale-true $N(\varphi_m = \pm 40°29')$. Shape-true at $\lambda=0$; $\varphi = \pm 40°29'$	" 209 (F136) BF 280
134		Cf footnote p re No 134				$y_\pi = F(x)$; $\pi\sin\varphi = \int_0^x F(x)dx$	b. F(x) acc to 104-109 pointing maps	F(x) acc. to 104. H: pairs of straight lines Collignon (1865) $m = 2\sqrt{\pi}$; $x = \sqrt{\pi}(1-\sqrt{2}\sin(\frac{\pi}{4}-\frac{\varphi}{2}))$; $y = \lambda\sqrt{\frac{2}{\pi}}\sin(\frac{\pi}{4}-\frac{\varphi}{2})$. Scale-true $N(\varphi_m = \pm 41°13'.3)$. Shape-true $\lambda=0$; $\varphi = \pm 41°13'.3$	TH 39, H165, G184, BF279
135								F(x) acc to 109. H: semi-ellipses Mollweide (1805) $m = \sqrt{8}:\pi$; $x = \sqrt{2}\sin\psi$; $y = \frac{\sqrt{8}}{\pi}\cos\psi$, where $\pi\sin\varphi = \sin 2\psi + 2\psi$. Scale-true $N(\varphi_m = \pm 45°46')$ Shape-true $\lambda=0$; $\varphi = \pm 45°46'$	TH41, ZB172; (F104), H162, G188; BF280
136							c. F(x) derived from 87 and ... Line-ring maps	Derived with same x. Neil (1890) x from $2\sin\varphi = x + \sin x$; $2y = \lambda(1+\cos x)$. Scale and shape true on base-line (Special case of 70)	Pet. M.H. 1890, p.43
137								Derived with same φ. Hammer (1900) $x = 2\varphi \cdot (\ldots)$; $2y = \lambda(1+\cos\varphi)$. " " " " " " "	" " 1900, p.41
138							d. F(x) acc to 119, 123	$mn = 4:(2+n_1\pi)$; $n_1 = \cos\varphi'$, x from $m\sin\frac{x}{m} + n_1 x = \frac{2}{n}\sin\varphi$, $y = \frac{n\lambda}{2}(\cos\frac{x}{m}+n_1)$. For true scale and shape, $\cos\varphi_m = \frac{n}{2}(\cos\frac{x}{m}+n_1)$	
139								Special case of 138. $m = 0.9046$; $n = 1.1054$, $n_1 = 2:\pi$; $mn = 1$. Centr. H and half base-line = 2.3426. Scale-true $N(\varphi_m = \pm 53°0'.5;\ x = \pm 1)$ True-shape at $\lambda=0$; $\varphi = \pm 53°0'.5$	referred to in K.H. Wagner, p.24
140							e. F(x) acc to 117, 130	$mn = 4:(3\sqrt{3}) = 0.7698$, x from $\sin\frac{x}{3m} = \frac{2}{3mn}\sin\varphi$; $y = n\lambda\sqrt{1-(\frac{2\sin\varphi}{3mn})^2}$. Generalization of 141. Scale true $N(\pm\varphi_m)$, where $\cos\varphi_m = n:\sqrt{4-3n^2}$; shape-true at $\lambda=0$; $\varphi = \pm\varphi_m$	
141	13							$m = n = \sqrt{0.7698}$ E by K.H. Wagner. Special case of 140. Centr. H and half base-line equally long. Scale-true $N(\varphi_m = \pm 53°41'.8)$. Shape-true at $\lambda=0$; $\varphi = \pm 53°41'.8$	K.H. Wagner p17 and 23
142								$n=1$; $m=0.7698$. Maurer. Special case of 140. Base-line scale- and shape-true everywhere	new
143							f. F(x) a parabola $F(x) = \sqrt{3\pi}(1-\frac{4x^2}{3\pi})$	Craster 1929, $x = \sqrt{3\pi}\sin\frac{\varphi}{3}$; $y = \lambda\sqrt{\frac{3}{\pi}}(1-\frac{4x^2}{3\pi})$. Scale-true $N(\varphi_m = \pm 36°45')$. Shape-true at $\lambda=0$; $\varphi = \pm 36°45'$	Deetz and Adams, p.165
144						B. $x = f(\varphi)$ prescribed; $y = \lambda\cos\varphi : \frac{dx}{d\varphi}$	f(φ) proportional to radius according to No. 2; $x = m\,tg\frac{\varphi}{2}$	General case. $y = \frac{2\lambda}{m}\cos\varphi\cos^2\frac{\varphi}{2}$; scale true $\varphi = \pm\varphi_m$, where $\cos^2\frac{\varphi_m}{2} = \frac{m}{2}$. Shape-true at $\lambda=0$; $\varphi = \pm\varphi_m$	new
145								Special case ($m=2$) of 144. Base-line scale- and shape-true	"
146								Prépetit-Foucaut (1862). Special case ($m=\sqrt{\pi}$) of 144. Centr. H and half base-line equally long. Scale-true $H(\varphi_m = \pm 35°32'.5)$. Shape-true at $\lambda=0$; $\varphi = \pm 35°32'.5$	TH44; H167; G181
147						C. Affine transformations from group 131-143	from 131-133	From 131. All H sine-lines $x_{147} = nx_{131}$; $y = \frac{1}{n}y_{131}$. Example $n = \sqrt{(\pi+2)}:2 = 1.134$ yields base line scale- and shape true	new
148								From 133. All H semi-ellipses; $x_{148} = nx_{133}$; $y = \frac{1}{n}y_{133}$. Example $n = \frac{1}{4}\sqrt{\pi(4+\pi)} = 1.184$ yields base-line scale and shape-true	new
149							from group 134-137	From 135. J. Bourdin. $x = nx_{135}$; $y = \frac{1}{n}y_{135}$. All H semi-ellipses. For $n = \pi:\sqrt{8} = 1.1107$, " " " " " "	TH 40 G190
150	14				b) All N scale similar; $y = n\lambda\cos\varphi$; pointing maps; Globe-biangles area-similar	World-map border ellipse of axis-ratio 1:2	$x = n\frac{\pi}{2}\sin\varphi$	$n=1$. All N scale-true. Shape-true at $\lambda=0$; $\varphi = \pm 50°27'.6$	
151								$n = \sqrt{8}:\pi$ Globe-biangles area-equal. No H scale true. Shape-true $\lambda=0$; $\varphi = \pm 50°27'.6$	
152								$n = 1:\sqrt{\pi}$ Fournier II (1646) The H $\lambda = \pm\frac{\pi}{2}$ are circles. " " " " . No H scale-true	TH 51

No	Fig.	Sub branch	Order	Class	Sub class	Group	Sort	Type	Literature

FAMILY II: STRAIGHT-SYMMETRICAL. BRANCH B: Not secondary-circle-spaced

No	Fig.	Sub branch	Order	Class	Sub class	Group	Sort	Type	Literature
53	17	A. ring-maps $F(\pm 90°)$ finite		All H arcs	For $\lambda^2 < \frac{\pi^2}{4}$ H are arcs thru $x=0$, $y=n\lambda$ and $y=0$, $x=\pm a$	For $\lambda^2 > \frac{\pi^2}{4}$ H thru 2 points	$x=\varphi$ Central H scale-true	$n=1$ Apianus I. y from $y^2+y(\frac{\pi^2}{4}-\lambda^2)\cdot\lambda=\frac{\pi^2}{4}-\varphi^2$ for $\lambda^2<\frac{\pi^2}{4}$; y from $\sqrt{\frac{\pi^2}{4}-\varphi^2}+\lambda-\frac{\pi}{2}$ for $\lambda^2>\frac{\pi^2}{4}$ Base line scale- and shape-true. Interval-true	TH 47
54							$a=\pi:2$	$n=2/3$ " " in the version y from $y^2(\frac{2}{2}\frac{\pi^2}{4}-\frac{2}{3}\lambda^2):\lambda=\frac{\pi^2}{4}-\varphi^2$ for $\lambda^2<\frac{9\pi^2}{64}$; $y=\sqrt{\frac{\pi^2}{4}-\varphi^2}+\frac{2\lambda}{3}-\frac{\pi}{2}$ for $\lambda^2>\frac{9\pi^2}{64}$	K.H. Wagner, p. 5; G 254 " "
55	20					$x=\pm a$; $y=n\lambda\cdot\frac{\pi}{2}$	$a=1$ $x=\sin\varphi$	$n=1$ Glareanus (1527) Base line scale and shape-true.	TH 48 G 254
56	20					For $\lambda^2>\frac{\pi^2}{4}$ as for $\lambda^2<\frac{\pi^2}{4}$	a " $x=\pi\, tg\frac{\mu}{2}$, where $\sin\mu=2\varphi:\pi$	$n=1$ van der Grinten III Bay line " " " ".	ZB 183 (Fig 112)
57	15	B. Prime-pt-map $F(\pm 90°)$ ...		(For further E of this sub branch see No. 215, 216)		$x=n\, tg\,\varphi$	$y=m\sin\frac{\lambda}{2}\cos\varphi$	Special E. H are curves IV order $y\sqrt{n^2+x^2}=mn\sin\frac{\lambda}{2}$, for $m=\pi$, entire length of all H as on globe; for $n=\varphi_a\cot\varphi_a$ intervals of $N(\varphi=\pm\varphi_a)$ from base-line as on globe.	new

FAMILY III. CURVED-SYMMETRICAL.

N not parallel straight-lines. Grid symmetrical to central-H and to base-line. $x(\lambda/\varphi)=-x(\lambda/\varphi)$; $y(\lambda/\varphi)=-y(-\lambda/\varphi)$; $x(\lambda/0)=y(0/\varphi)=0$

BRANCH A ROW CIRCULAR. ALL H NOT CONCENTRIC ARCS WITH CENTER-POINTS ON CENTRAL-H.

$y^2+(\Phi+r-x)^2=r^2$, WHERE r AND Φ ARE FUNCTIONS OF φ. FOR $\varphi=0$; $\Phi=\frac{1}{r}=0$

No	Fig.	Sub branch	Order	Class	Sub class	Group	Sort	Type	Literature
58	72	A. 2 N intersect. As φ increases Φ increases and r decreases	Prime point maps For $\varphi=\pm 90°$ $y=r=0$ $\Phi=\pm a$	I. circle-grids of first class: $x^2+(y-g\lambda)^2=g^2$, g and Λ functions of λ.	a) central H scale true $\Phi=\varphi$ $g=\frac{1}{2\lambda}(\frac{\pi^2}{4}+\lambda^2)$ $a=\frac{\pi}{2}$	A. Base line scale-true. All H arcs through $(y=0, x=\pm\frac{\pi}{2})$ and $(x=0, y=\lambda)$ $g=\frac{1}{2\lambda}(\frac{\pi^2}{4}+\lambda^2)$	a. r determined by dividing-points on 2 H $\pm\lambda_0$ and on $\lambda=0$	$\lambda_0=\pm\frac{\pi}{2}$; the 2 H equal-spaced. Globular E by Nicolosi (1660). $2r(\frac{\pi}{2}\sin\varphi-\varphi)=\frac{\pi^2}{4}+\varphi^2-\pi\varphi\sin\varphi$; $2g(\frac{\pi}{2}\sin\lambda-\lambda)=\frac{\pi^2}{4}+\lambda^2-\pi\lambda\sin\lambda$	TH 47; ZB 101; H 148; G 253, G 284
59								$\lambda_0=\pm\pi$, " " " " . Foil-globular E reys et H-lines λ_0 $y(\pm\pi)=5\pi:8$. Dividing points on these H-lines: $x(\pm\pi)=\frac{5\pi}{8}\sin\varphi$; $y(\pm\pi)=\pm\frac{\pi}{8}(3+5\cos\varphi)$	
60							r determined by dividing points on $\pm\lambda_0$ and the points $x=0$, $y=\lambda$	$\lambda_0=\pm\pi$ Apparently globular E Maurer. Dividing points on H lines λ_0 $x(\pm\pi)=\frac{5\pi}{8}\sin\frac{3\varphi}{4}$; $y=\pm\frac{\pi}{8}(3+5\cos\frac{3\varphi}{4})$	Z.d.Ges.f.E. 1922, p. 124, I " " " " " " II
61								$\lambda_0=\pm\pi$. Van der Grinten's E Dividing points on H-lines λ_0 acc. to equation of these H-lines $x^2+(y-\frac{3\pi}{8})^2=(\frac{5\pi}{8})^2$ and equation $y+\frac{\pi}{4}=x\frac{\pi-4\sqrt{10}}{4}$	" " " " " III ZB 185 (F114); BF 284
62	72						b. Pr circular (paneconical) $r=\cot\varphi$	Paneconical circle-grid Base line and central-H scale-true. Not rectangular.	
63		–					c. $r^2=y^2+(\Phi+r-x)^2$	$r=\frac{1}{2\varphi}(\frac{\pi^2}{4}-\varphi^2)$ Rectangular circle-grid. Base-line and central-H scale-true. Not paneconical	TH 116
64						B Base-line not scale-true, cell-equal	a. H-lines $\pm\lambda_0$ equal-spaced	$\lambda_0=\pi$ Dividing points on H-lines $\lambda_0=\pm\pi$: $x(\pm\pi)=1.673\sin\frac{2\varphi\alpha_0}{\pi}$; $y(\pm\pi)=\pm 1.673(\cos\frac{2\varphi\alpha_0}{\pi}-\cos\alpha)$, where $\alpha_0=110°7'=1.9219$	new
65						$g=\pi(2\sin\alpha)$ $\alpha=\sin^{-1}(\frac{2\lambda}{\pi}+\frac{\sin 2\alpha}{2})$	b Zones between the N are cell-equal.	Equations in the text.	new
66	72				b) central-H not scale-true	A. Base-line scale true $\Lambda=\lambda$	World map border in circle $a=\pi$	$\Phi=\pi\, tg\frac{\mu}{2}$, where $\sin\mu=2\varphi:\pi$; r according to special geometric prescription Van der Grinten I	ZB 182 (F110)
67								" " " " " ; $r=\cot\mu$. Rectangular circle-grid. " " " II	" 183 (111)
68									

SNo	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
168					–	B Base-line not scale-true	a. area-equal	Central-H half so long as base-line. Zones between the N area-equal. $a = 1.1255$, $\alpha_0 = 126°52'$; ...	(F 65)
169	67						b shape-true circle-grids $x = n\lambda$	$n = \sqrt{2}$. Lagrange (1797). Change of length ratio in $\varphi = \lambda = 0$ the slowest. Map larger than map plane [Ostwald's Klassiker No 55, p. 56]	TH 123; H 141; G 219, BF 283
170	66						$tg\left[\frac{\pi}{4} - \frac{\rho}{2}\right] =$	$n = 1$. All-circular E, transverse-axical, map covers map plane exactly	TH 120; H 145; G 228
171	}						$tg^n\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)$	$n = 0.7$. Base-line about double so long as central-H. Map finite.	new
172	} 67						$\rho = a \operatorname{cosec} \alpha$	$n = 0.6115$. Border-meridian-length : equator-length = equator-length : central-H-length = 1.4303 : 1. Map finite.	TH 121
173	}						$r = a \cot \beta$	$n = 0.5$. Lambert (1772). Base-line just as long as central-H. [Ostwald's Klassiker No. 54, p. 34, fig. 11]	ZD 180 (F 103), H 266, G 229
174	72						c. H for $\lambda = \pm 90°$ equal-spaced	Neil's intervening circle-grid Φ and Λ centers from No 158 and 170; $\rho = (1 - \Lambda^2) : 2\Lambda$, where $2\Lambda = \frac{2\lambda}{\pi} + tg\frac{\lambda}{2}$; $2\Phi = \frac{2\varphi}{\pi} + tg\frac{\varphi}{2}$	TH 137; ZD 102; BF 284
175	72			II. Not all H are circles	Central-H scale-true $\xi = \varphi$	A. Base-line scale-true $\Lambda = \lambda$	a. N for $\lambda = \pm 90°$ equal-spaced semi-circles	Fournier I (1646) H-Lines as in Mollweide (No 135) N-lines circles through the 3 dividing-points on $\lambda = 0, +\frac{\pi}{2}, -\frac{\pi}{2}$. Globe-biangles area-equal	TH 115, H 147, G 255
176	62						b. all-conical (panconical) $r = \cot \varphi$	"Ordinary polyconic E" of American Coast and Geodetic Survey (1855). All N scale-true. $\beta = \lambda \sin \varphi$. Shape-true on $\lambda = 0$.	TH 114, ZD 145; (F 40); H 135; G 143, BF 281
177	72							"Rectangular polyconic E" of British War Office (1860) $tg\frac{\beta}{2} = \frac{\lambda}{2}\sin\varphi$. All N and H intersect rectangularly.	TH 111; ZD 148; H 135; G 143; BF 281
178	62						c. Every N = arc thru the 3 points ($y = 0$, $x = \varphi$) and $y = \pm\frac{\pi}{2}(3 - 5\cos\varphi)$, $x = \frac{5\pi}{4}\sin\varphi$	All N equal spaced. Secondary-circle-spaced all-globular. $tg\frac{3}{2}\tau = \frac{\varphi \cdot 2\pi}{y\pi}$; $\beta(\lambda) = \frac{\lambda}{\pi}C_\pi$	new
179	64					B Base-line not scale-true	all-conical (panconical)	Area-true pan-conical E. Maurer. β from $\beta = \sin\beta\cos^2\varphi + \lambda\sin^3\varphi$. True-shape at $\varphi = \pm\pi/2$	TH 112; H 146
180	72		b. Polar-line maps $\rho > 0$ for $\varphi = \pm\frac{\pi}{2}$	Circle-grid of first class as 158-174	Central H scale-true	Base-line scale-true $\Lambda = \lambda$	$\rho = (\pi^2 + \lambda^2) : 2\lambda$	Maurer $r = \pi \cot \mu$, where $\mu = 2a\pi\, tg(\varphi : \pi)$. Grid as 167 but N lines numbered differently.	new
181	67	B. All N thru 2 points ($\varphi = 0$, $\lambda = \pm\pi$) $r = (\pi^2 + \lambda^2) : 2\lambda$	Polar-line map	circle-grid of second class $x^2 + (y - \rho - \Lambda^2) = \rho^2$	rectangular $\rho = \frac{\Lambda^2 + \varphi^2}{2(\varphi - \Lambda)}$	Shape-true. $\Phi = \operatorname{cosec}\beta - \cot\beta$, where $\beta = n \log nat \cot\frac{\delta}{2}$	$\Lambda = \operatorname{cosec}\alpha - \cot\alpha$, where $\log nat \cot\frac{\alpha}{2} = n\lambda$	$n = 1$ Grids as 169-173, but N- and H-lines exchanged and should be noted 2λ rather than φ and φ rather than 2λ	TH 127

FAMILY III. Curved-symmetrical. Branch B - not rowcircular

SNo	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
182	69	A. Transformed from family I (transverse-axical polar-grid) or from family II or family III branch A	a. Affined E. From x and y of E. from which transformed $x_1 = mx$, $y_1 = ny$	I. Transformed from polar-grid of transverse-axical radial projections	Transformed from prime point map	A. Original E. plane-perspective.	Original E. inner-perspective.	Original E (projection from which transformed) center-perspective (No. 1) transverse-axical. Maurer. $x_1 = m \lg \varphi \sec \lambda$, $y_1 = n\, tg\, \lambda$. All great-circles straight-lines. Shape-true at $x_1 = 0$; $y_1 = \pm \sin L$, if $\cos L = n : m$, in $y_1 = 0$; $x_1 = \pm \sin L$, if $\cos L = m : n$ (C.f. Z.f. Vermessungswesen 1922, p. 14)	Pet. M.H. 1914, II, p116
183	64					B. Original E not plane-perspective	Original E area-true	Original E Lambert (No. 23) transverse-axical. Maurer $m = 1.0056$; $x_1 = 2m \sin\frac{\delta}{2}\cos\alpha$; $y_1 = \frac{2}{m}\sin\frac{\delta}{2}\sin\alpha$, where $\cos\delta = \cos\lambda\cos\varphi$ and $tg\,\alpha = \sin\lambda\cot\varphi$; $n = 1 : 1.0056$; shape-true at $\lambda = 0$; $\varphi = \pm 15°$	new
184	64			II. Original E row-circular	Original E. Central-H scale-true	Original E. base-line not scale-true	Original E panconical	Original E. area-true (No. 179) Maurer $n = 1.2743$; $m = 1.2743$; $x_1 = \frac{1}{n}(\varphi + 2\cot\varphi\sin^2\frac{\beta}{2})$; $y_1 = n \cot \varphi \sin\beta$; $\beta = \sin\beta\cos^2\varphi + \lambda\sin^3\varphi$. Shape-true at $\varphi = 0$; $\lambda = \pm 116° 19'$	new

SNo	Fig.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
195	72		B. Altitude from $x(\varphi/\lambda)$ and $y(\varphi/\lambda)$ of the orig. E result $x_1(\varphi/2\lambda) = m(\varphi/\lambda)$, $y_1(\varphi/2\lambda) = 2ny(\varphi/\lambda)$	Orig. grid quasi grid transverse axical, radial F	Orig. E perimeter-point map	Orig. E not plane-perspective	Orig. E interval-true	Original E. Mercator-Postel No. 24. Aitoff's <u>Planisphere</u>. $m = n = 1$; $x_1(\varphi/2\lambda) = x(\varphi/\lambda)$; $y_1(\varphi/2\lambda) = 2y(\varphi/\lambda)$; where $\cos\sqrt{x^2+y^2} = \cos\lambda\cos\varphi$; $y = x\sin\lambda\cot\varphi$. Base-line and central-H scale-true. Shape-true at $\varphi = \lambda = 0$	ZB 176; BF 250
196	64						Orig. E area-true	Original E Lambert No. 23. <u>Hammer's</u> Planisphere (1892) $m = n = 1:\sqrt{2}$; $x_1(\varphi/2\lambda) = x\frac{(\varphi/\lambda)}{\sqrt{2}}$; $y_1(\varphi/2\lambda) = \sqrt{2}y(\varphi/\lambda)$; $x^2+y^2 = 4\sin^2\frac{\varepsilon}{2} = 2(1-\cos\varphi\cos\lambda)$; $y = x\sin\lambda\cot\varphi$. Shape-true at $\varphi = \lambda = 0$	ZB 176 (F 106); BF 280
197 }	72		C. Meridian-spacing of original E altered.	All H equal-spaced.	Orig. E circle-grid of first class.	Orig. E central-H scale-true $\Phi = \varphi$	Orig. E. base-line scale-true $\Lambda = \lambda$	Orig. grid full-globular No 159. <u>Maurer</u> primary-circular-all-globular. All H equal-spaced. $\varrho = (\pi^2 + 4\lambda^2):(8\lambda)$, $x = \varrho\sin\frac{2\varphi z}{\pi}$; $y = \lambda + 2\varrho\sin\frac{2\varphi z}{\pi}$, where $\sin z = \pi:(2\varrho)$	Z.d. Ges. f. Erdk. 1922, p. 125
198 }					Orig. E cylinder-circular	Orig. E central-H not equal-spaced	Orig. E area-true	Orig. E Mollweide No. 135. G.G. <u>Schmidt</u> (1803). All H equal-spaced. $x = \sqrt{2}\sin\psi_1$; $y = \frac{2\lambda\sqrt{2}}{\pi}\cos\psi_1$; where $\int_0^{\psi} u\,d\psi$ and $u^2 = 2\cos^2\psi + \frac{8\lambda^2}{\pi^2}\sin^2\psi$. Not area-true.	TH 146; H 164 footnote
189	65	A Shape-true	N and H algebraic curves.	N and H curves of second order.		N-Ellipses	N-Ellipses confocal	Azimuth-equal map <u>Littrow-Maurer</u> (1833, 1905) H confocal hyperbolae. $x = \operatorname{tg}\varphi\cos\lambda$; $y = \sec\varphi\sin\lambda$, All straight lines on map are azimuth-equals.	TH 151, ZB 221 (F 152), H 255
190	67					N-hyperbolae	N-hyperbolae	<u>Fiorini</u> (1881) $x = \sec\lambda\sin\varphi$; $y = \operatorname{tg}\lambda\cos\varphi$	TH 151, H 255
191				N and H of higher order		N curves of fourth order	N-lemniscates	Transformed from 189 by reciprocal rays. $x = \sin\varphi\cos\varphi\cos\lambda : D$; $y = \cos\varphi\sin\lambda : D$, where $D = 1 - \cos^2\lambda\cos^2\varphi$. Azimuth-equals are circles. Maurer	Marine-Luftflotten-Rdschau 1933
192	67					H evolvents of epicycloids		<u>August</u> epicycloidal E (1874) $x_1 + iy_1 = \frac{\rho}{2}(3-\rho^2)$, $\rho = (1+\sqrt{1-Z^2}):Z$; $Z = x+iy$; $x = \sin\varphi : D$, $y = \cos\varphi\sin\lambda : D$, where $D = 1+\cos\lambda\cos\varphi$. Border meridian an epicycloid.	TH 153
193			N and H are transcendent curves					<u>Eisenlohr</u> (1870) $x + iy = 2i(v+iu) + \sqrt{2}\left[e^{v+iu} - e^{-(v+iu)}\right]$, where $\operatorname{tg} u = \sin\frac{\varphi}{2} : \left[\cos\frac{\varphi}{2} + \cos\frac{\lambda}{2}\sqrt{2\cos\varphi}\right]$; $v = \frac{1}{2}\ln\left[\frac{\cos\frac{\varphi}{2} + \sqrt{\cos\varphi}\cos(45^\circ - \frac{\lambda}{2})}{\cos\frac{\varphi}{2} + \sqrt{\cos\varphi}\cos(45^\circ + \frac{\lambda}{2})}\right]$	Crelle's Journal, Vol. 72, 1870, p. 143.
194			N and H by double-periodic elliptic integrals of the first class					<u>Guyou</u> (1887) $x = \int_0^{\varphi_e}\frac{d\varphi_e}{\sqrt{1-\frac{1}{2}\sin^2\varphi_e}}$, where $\sin\varphi = \sin\varphi_e\sqrt{1-\frac{1}{2}\sin^2\lambda_e}$; $\cos\varphi\sin\lambda = \sin\lambda_e\sqrt{1-\frac{1}{2}\sin^2\varphi_e}$. Annales hydrographiques, Paris 1887, p. 16	Deetz v. Adams, p. 160, (F. 74)
195	72	C. All H parallel straight lines	Shape-true on base-line and central H	cylinder-radial		double-cylinder-radial	base-line scale-true	Central-H scale-true $\cot x = \cot\varphi\cot\lambda$, $y = \lambda$. All great-circles perpendicular to base-line or to central-H are straight-lines.	Ann. d. Hydr. 1919, p. 20
196	70		Shape-true at two equator-points. (2 further E of this sub-branch as special cases of No 199 and 200)			The azimuth-equals of these 2 points are straight-lines		<u>Maurer</u> (1919) double-counter-azimuthal E. Shape-true points $\varphi = 0$; $\lambda = \pm\lambda_1$; $x = 2\sin\varphi(\cos^2\lambda_1 - \sin^2\lambda):\sin 2\lambda_1$; $y = \sin 2\lambda : \sin 2\lambda_1$	

FAMILY IV: Less regular. Neither true-circular nor double-symmetrical

BRANCH A: Grid symmetrical to central-H but not to base-line

SNo	Fig.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
197	33	A cylindro-radial $y = g(\lambda)$ All H parallel straight lines	a. H straight lines equal spaced $y = n\lambda$	I. Pole-point maps	a) cylindrical: cylinder-perspective. N parallel straight lines $x = f(\varphi)$; eye-point below base-circle $q > 0$	a. intersecting cylindrical		$n = \cos\varphi_m$, $x = \cos\varphi_m[\sin\varphi + q(1-\cos\varphi)] : \cos\varphi$; $y = \lambda\cos\varphi_m$; φ_m scale-true; $\varphi = 0$ shape-true. (For $\varphi = 0$ belonging to family II, Branch A, Subbranch A, No. 92, Fig. 4)	
						b. contiguous-cylindrical		$n = 1$ $x = \operatorname{tg}\varphi + q(1-\cos\varphi) : \cos\varphi$; $y = \lambda$. On $\varphi = 0$ scale and shape-true (For $\varphi = 0$ belonging to " " " " ")	

SN.	FN.	Sub-branch	Order	Class	Sub-class	Group	Sort	Type	Literature
199	85				b) A shape-true pt S on central m.	Base-line circle: true n=1	a. Gont-circles thru S straight lines	<u>Schay</u>'s radial-directional E (1913). $y=\lambda;\ x=\lambda(\cos\varphi_0\, tg\,\varphi - \sin\varphi_0\, tg\,\lambda)\cdot\sin\lambda$ (For $\varphi_0=0$ belonging to family III, branch B, subbranch C, FNo. 77)	Ann. d. Hydr. 1913, p.
200					$(\lambda=0;\ \varphi=\varphi_0)$		b. Az. m. the equals of the direction point S are straight lines.	<u>Craig</u>'s retroazimuthal E (1909) $y=\lambda,\ x=\lambda(\sin\varphi\cos\lambda-\cos\varphi\, tg\,\varphi_0):\sin\lambda$. Counter-azimuthal. (For $\varphi_0=0$ belonging to family III, branch B, subbranch C, FNo. 70)	ZB 218 (F150)
201, 202	75			II. Polar-line maps	All N parallel arcs $x=f(\varphi)+r\sqrt{r^2-n^2\lambda^2}$	A interval-true $f(\varphi)=\varphi$	a. contiguous cylindrical b. intersecting cylindrical	$n=1,\ y=\lambda,\ x=\varphi+r-\sqrt{r^2-\lambda^2}$. All N greater than circumference of globe. Shape-true at points $\varphi=\lambda=0$. $n<1;\ y=n\lambda,\ x=\varphi+r-\sqrt{r^2-n^2}$. Length of all N = L = $2r\alpha$, where $\sin\alpha=n\pi:r$; $L=2\pi\cos\varphi_m$ for $\cos\varphi_m=\frac{r\alpha}{\pi}=\frac{n\alpha}{\sin\alpha}$. Shape-true at $\lambda=0;\ \varphi=\pm\varphi_m$	
203, 204	77					B. area-true $f(\varphi=\sin\varphi)$	a. contiguous cylindrical b. intersecting cylindrical	$n=1;\ y=\lambda,\ x=\sin\varphi+r-\sqrt{r^2-\lambda^2}$. All N greater than circumference of globe. Shape-true at $\varphi=\lambda=0$ $n<1,\ y=n\lambda;\ x=(\sin\varphi+r\sqrt{r^2-n^2\lambda^2}):n$. Length of all N = L = $2r\alpha$, where $\sin\alpha=n\pi:r$; $L=2\pi\cos\varphi_m$ for $\cos\varphi_m=\frac{r\alpha}{\pi}=\frac{n\alpha}{\sin\alpha}$. Shape-true at $\lambda=0,\ \varphi=\pm\varphi_w$, where $\cos\varphi_w=n$.	
205	73		b. meridians lines not equal-interval	Polar-line maps	All N equal-spaced	All N parallel arcs $y=r\sin\frac{\lambda\alpha}{\pi}$; $x=f(\varphi)+(1-\cos\frac{\lambda\alpha}{\pi})$ a and r constant	a. interval-true $f(\varphi)=\varphi$	Type according to values α and r. Scale-true N (= φ_m), where $\cos\varphi_m=r\alpha:\pi$; $y=r\sin\frac{\lambda\alpha}{\pi}$; $x=\varphi+r(1-\cos\frac{\lambda\alpha}{\pi})$.	
206	74						b. zone-true	" " " " " " " " " , where $\cos\varphi_m=r\alpha:\pi$, $y=\pi\sin\frac{\lambda\alpha}{\pi}:\sin\alpha$; $x=\sin\varphi+2\pi(\sin\frac{\lambda\alpha}{2\pi})^2:\sin\alpha$	
207		B not cylinder-radial	a. cylinder-circular $x=f(\varphi)$	I secondary-circle-spaced $y=\lambda g(\varphi)$	A. H-straight lines thru point $y=0,\ x=p$	Polar-point maps Point (y=0; x=P) picture of a pole	a. Picture of picture of both poles	Loxodrome map. $P=h\,cosec\,\beta$, $x=Mh:[h\cos\beta+M\sin\beta]$; $y=\lambda(1-\frac{x}{h}\sin\beta)$; β and h constants. M = lognat $tg(\frac{\pi}{4}+\frac{\varphi}{2})$. Perspective of the Mercator map from point h perpendicular over point $\varphi=\lambda=0$ on plane below angle β thru equator.	
208	50						b Pt. P picture of only one pole	interval-true $P=\frac{\pi}{2}$, $x=\varphi$; $y=\lambda(\pi-2\varphi):\pi$ as in No. 105; but same equat. applies for both hemispheres.	
209	53							not interval-true x and y as in No. 104 and 106, " " " " " " "	
210, 211	50					B. Polar-line maps No pole shown as point.	a. interval-true $P=\frac{\pi}{2}:x=\varphi$	$y=\lambda(\pi-\varphi):\pi$ as in no. 114; " " " " " " " $y=\lambda[(\varphi-\varphi_2)\cos\varphi_1-(\varphi-\varphi_1)\cos\varphi_2]:(\varphi_2-\varphi_1)$ as in No. 125; " " " " " " "	
212, 213	51						b. area-true	$x=m\pi-\sqrt{m^2\pi^2-2\pi\sin\varphi}$; $y=\lambda(m\pi-x):\pi$, where $m=\sqrt{8}:3\pi$ as No. 132, but same equat. " " " " $x=\sqrt{\pi}[1-\sqrt{2}\,\sin(\frac{\pi}{4}-\frac{\varphi}{2})]$, $y=\lambda\sqrt{\frac{2}{\pi}}\sin(\frac{\pi}{4}-\frac{\varphi}{2})$, where $m=2:\sqrt{\pi}$ as No. 134, " " " " " "	
214	53						c. neither interval-true nor area true	x and y as No. 112, 115, 124 and 126, " " " " " "	
215, 216	58, 60			II. not secondary circle spaced	One shape-true point S on central M	Polar-point-maps.	Azimuth equals of the direction-point S are straight lines	Interval true $x=\varphi-\varphi_0$; $y=-x\, tg\, A$, where $\cot A=(\cos\varphi_0\, tg\,\varphi-\sin\varphi_0\cos\lambda):\sin\lambda$. Counter-azimuthal. (For $\varphi_0=0$, belongs to fam II, branch B, Subbr. D, FNo. 16) Not interval true, e.g., $x=m\sin\frac{\varphi-\varphi_0}{2}$, $y=-x\, tg\, \alpha$, $A=(\cos\varphi_0$ etc. same as above " " . (" " " " " " " " 18)	
217, 218	79		c. not cylinder-circular	I. Shape-true	a) δ_m straight $\delta_m > \frac{\pi}{2}$	Row-circular. All N are arcs	Circle-grid of the first class	<u>Lambert and Lagrange</u>. $tg(\frac{\pi}{4}-\frac{z}{2})=(tg\frac{\delta}{2}:\cot\frac{\delta_m}{2})^n$; N r = $\cot\beta$, center ($y=0$, $x=cosec\,\beta$); H-circle ρ = $cosec\; n\lambda$, center ($x=0$; $y=-\cot n\lambda$)	TH 129 (F68); H 261
					b) Grid also symm. to $\lambda=\frac{n\pi}{4}$	H and N by double-periodic elliptical integrals of class I.		<u>Peirce</u> (1879). Quincuncial E. Polargrid of the transverse-axial E No. 194. cos at $(x+iy+K;K)=tg\frac{\delta}{2}(\cos\lambda+i\sin\lambda)$, where $k=1:\sqrt{2}$; $K=\int_0^1\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$	TH 162; H 267 (F63)
219	76			E. One shape-true pt. $S(\varphi_0)$ on centr. M.	Azimuth equals of direction points S are straight lines	Secondary lines of point S drawn so that true-area occurs.		<u>Maurer</u> Area-true counter azimuthal E. $\frac{x}{y}=tg\,\varphi_0\cos\varphi\; cosec\,\lambda-\sin\varphi\cos\lambda$ (For $\varphi_0=0$ belongs to family II, branch B, new subbranch D, FNo 63)	Pet. Mitt. 1914 II, p.67

No	FNo	Sub-branch	Order	Class	Sub-class	Group	Sort	type	Literature
					FAMILY IV.		BRANCH B: Grid not symmetrical to central H		
0	79	Polar-point maps	All N and H are circles	Circle-grids of first class	Shape-true	North and South poles separated	$0<n<1$	Grid symmetrical to base-line $\varphi = 0$. Shape-true circle-grid as No. 171-173; but lateral world map, i.e., $(\lambda + \gamma)$ employed rather than λ.	
21	79	—	—	—	—	—		" not " " " " " " " " as No. 217, " " " " " " " " " ".	
2	79					North and south poles at same point.	$n = 0$	Grid symmetrical to base line. Obtained from Mercator map by reciprocal rays. Transformation center at $x = 0;\ y > \pi$. All loxodromes are circles.	A.d. Hyd. 1919, p. 217 (F3)
3	79							" not " " " " " " " " " " " at $y \gtrless 0;\ > \pi$ " " " "	
4	40	—	All N and H are straight lines	All loxodromes are straight lines	cylinder-circular	secondary circle-spaced	straight-celled	Mercator map parallel projected onto plane, parallel neither to equator nor to central-meridian. H convergent straight lines. $x = c \lg nat\ tg\left(\frac{\pi}{4} + \frac{\varphi}{2}\right);\ y = m\lambda + ny$	
5	32	—	—	—	not cylinder-circular	not secondary-circle spaced	not cellular	" " projected " " " " " " " to central-meridian. Sighting-pt. in the finite. N and H convergent straight lines.	
					FAMILY V.		COMBINATION GRIDS: equations different for different parts of world map.		
								BRANCH A: Double-symmetrical. Equator a straight line	
26		Uniform rule for N	All N parallel straight lines	All H pairs of straight lines				Here belong projections listed in family II under No. 104-106, 113-115, 124-126, 132, 134.	
27		Uniform rule for H	All H circles as in circle-grid of class I	All N constructed from arcs	H as in SNo 190			Maurer's Isogone Map. All N of each four arcs with no breaks. H-equation: $x^2 + \left[y - tg\left(\frac{\pi}{4} - \frac{\lambda}{2}\right)\right]^2 = \sec^2\left(\frac{\pi}{4} - \frac{\lambda}{2}\right)$ N of inner hemisphere $(x - \operatorname{cosec} \varphi)^2 + y^2 = \cot^2 \varphi = r_1^2$; 2 connecting arcs with radius $r_2 = tg\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)$ and closing semi-circle with radius $r_3 = 2r_2$.	Pet. Mitt. 1911, II, p. 91
								BRANCH B: Not double-symmetrical. Equator not a straight-line	
8	—	Uniform rule for H. True-circular. $r = f(\delta)$	Star projections. Northern hemisphere closed. South hemisphere slit in n lobes acc. to border meridian. N of north hemisphere are straight lines	N of south hemisphere are straight lines.	Center-interval-true $r = \delta$	north hemisphere radial	southern lobes not equal	$n = 8$ Petermann (1865) Border-meridians 10°, 60°, 100°, 155° E. and 35°, 55°, 120°, 155° W. (Pet. M. Erg., No. 16, p. 67)	TH 144
9	—						southern lobes equal	$n = 5$, Berghaus (1879) " " 56°, 128° E.; 16°, 88°, 160° W.	TH 142
20						north hemisphere conical	southern lobes equal	$n = 4$ Steinhauser's conoclatic E. Border-meridians 0° ± 40°, 180°. World map in sector of 240° angle-opening	TH 142
1				H of southern hemisphere are curves	area-true	northern-hemisphere radial	southern lobes equal	$n = 6$. northern-hemisphere as SNo. 23, southern hemisphere $\delta > \frac{\pi}{2}$, $r = 2\left(\sqrt{2} - \cos\frac{\delta}{2}\right)$; $a = \lambda \cos\frac{\delta}{2} : \left[\sqrt{2} - \cos\frac{\delta}{2}\right]$, a and λ from straight-line central-H. Border-meridians 40°, 100°, 160° E.; 20°, 80°, 140° W.	

S	o	File	Sub-branch	Order	Class	Subclass	Group	Sort	Type	Literature
.	2		Rule not uniform for N or for N	Star-E. North-pole Center-pt. Southern hemisphere slit into N lobes according to straight border meridians.	Equator an n-angle. Each side a symmetry-line of the globe-cell it halves	Each N a pair of straight-lines. N of each lobe are parallels to their pieces of equator.	All N equal-spaced.	Equator irregular n-angle	$n = 8$ Jäger (1865) Border-meridians as No. 228 (Pet. M. Erg. Heft 16, p. 67)	TH 142
.	3						Border-meridian scale-true	Equator regular n-angle.	$n=6$ Equator and border-meridians: 40° 100° 160° E.; 20°, 80°, 140° W	
.	4						All N equal-spaced. All N unequal-spaced. Area-true	Equator regular n-angle	$n = 4$ For $-\frac{\pi}{4} < \lambda < +\frac{\pi}{4}$ and $\delta < \frac{\pi}{2}$, $y = \sqrt{\pi} \sin\frac{\delta}{2}$, $x = 4\lambda : \sqrt{\pi}$. Neighboring globe-eighths symmetrical to border-line. Border-meridians as 230	
.	5					Equator a square. Border meridians as 2?0, [illegible]	Equal-area of each globe-eighth	Each N constructed of 4 quadrants and straight-lines	$n = 4$. In the first quarter of the northern hemisphere. Here straight-lines $\operatorname{tg}\alpha = 4\lambda : \pi$. x_0 and y_0 are independent, for the N, from δ according to equat. $y_0^2 - \left(\frac{\sqrt{\pi}}{2} - x_0\right)^2 = \frac{\pi}{2}$ and $\pi \sin\frac{2\delta}{2} = 2x_0 y_0 - x_0^2 + (y_0 - x_0)^2 \frac{\pi}{4}$. Then, for $x < x_0$, $y = y_0$ and, for $x > x_0$, $(x - x_0)^2 + (y - x_0) = (y_0 - x_0)^2$.	
2	6			Polyhedron-projections. Continuation onto an n-plane around the globe	Perspective reproduction onto each face of the n-plane	Perspective from center of globe	n a small number	regular n-plane	Reproduction on cube, which is contiguous with globe at ($\varphi = 0°$, $\lambda = 10°$ W and 170° E), ($\lambda = 80°$ E, $\varphi = 60°$ N and 30° S), ($\lambda = 100°$ W, $\varphi = 30°$ N and 60° S)	Deetz and Adams, p 62, (F46)
								Irregular n-plane	Reproduction on rectangular eight cornered box. Floor and lid are contiguous with globe at $\varphi = \pm\frac{\pi}{2}$. 4 side-walls parallel to meridian-planes $\lambda = 20°$ E and 70° W, distance $1\frac{1}{2}$ from center of uniform globe. Grid on floor and lid is center-perspective, earth-axical, on sides $y = 1.5 \operatorname{tg}\lambda$; $x = 1.5 \sec\lambda \operatorname{tg}\varphi$ Maurer	new

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13. ABSTRACT

This work is an edited translation of Maurer's Ebene Kugelbilder, Berlin, 1935. It contains an exhaustive classification system for map projection and a catalogue of 237 named projections arranged in classes and various subclasses on the basis of such characteristics as properties (conformality, equivalence, etc.), the appearance of the latitude-longitude grid and the form of the equations (linear, algebraic transcendental, etc.). It is believed that this translation will help fill a very serious gap in the literature of English-speaking geographers.